A continuous super-Brownian motion in a super-Brownian medium*

Donald A. Dawson† and Klaus Fleischmann‡

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Abstract

A continuous super-Brownian motion X^{ϱ} is constructed in which branching occurs only in the presence of catalysts which evolve themselves as a continuous super-Brownian motion ϱ. More precisely, the collision local time $L[W,\varrho]$ (in the sense of Barlow et al. [BEP91]) of an underlying Brownian motion path W with the catalytic mass process ϱ governs the branching (in the sense of Dynkin's additive functional approach). In the one-dimensional case, a new type of limit behavior is encountered: The total mass process converges to a limit without loss of expectation mass (persistence) and with a positive (finite) limiting variance, whereas starting with a Lebesgue measure ℓ, stochastic convergence to ℓ occurs.

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†Department of Mathematics and Statistics, Carleton University, Ottawa, Canada K1S 5B6. E-mail: donald_dawson@carleton.ca

‡Weierstrass Institute for Applied Analysis und Stochastics (WIAS), Mohrenstr. 39, D–10117 Berlin, Germany. E-mail: fleischmann@iaas-berlin.d400.de

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1 Introduction

1.1 Motivation

The motivation for the study of *catalytic branching* comes from two different viewpoints of catalytic chemical and biological systems – one microscopic and the other macroscopic. At the *microscopic* level in a chemical reaction a molecule can be involved in certain chemical reactions only in the presence of a catalyst. At the *macroscopic* level spatially distributed chemical reactions are described by reaction diffusion equations and the catalyst enters as a spatially heterogeneous rate function. In some cases the catalyst may only be present in some *localized regions* such as networks of filaments or the surfaces of pellets.

Figure: Enzymes located on a network of filaments

Such a system is modelled by a *catalytic reaction diffusion equation* in R^d of the form

$$-\frac{\partial u}{\partial s} = \frac{1}{2}\Delta u + \varrho_s R(u), \qquad 0 \leq s \leq t, \tag{1}$$

with *terminal condition* $u\big|_{s=t} = \varphi$, where R is the reaction term and ϱ_s describes the spatial density of the catalyst at time s and is assumed to be given by a continuous measure-valued path $s \mapsto \varrho_s$. (From a probabilistic point of view, this *backward* formulation is more natural than the usual forward one where φ would be the initial function.) Since ϱ_s might be rather singular, the precise meaning of this equation is made clear in the related *evolution form (integrated form)*, namely

$$u(s,t,a) \quad = \quad \int \mathrm{d}b\, \mathrm{p}(t-s,b-a)\,\varphi(b) \\ \qquad + \int_s^t \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \mathrm{p}(r-s,b-a)\, R\big(u(r,t,b)\big). \tag{2}$$

where $\mathrm{p}(t,b)$ denotes the transition density of standard Brownian motion in R^d.

Partial differential equation methods has been used to study (1) in the case in which ϱ_s is assumed to be a well-behaved measure concentrated on an open set or on a hypersurface; cf. Chadam and Yin [CY94], Chan and Fung [CF92], Bramson and Neuhauser [BN92], Durrett and Swindle [DS94]. On the other hand in some *biochemical reactions* (e.g. glycolysis) enzymes serving as catalyst are located on a filament network which might be modelled by a fractal set. (See the figure to get an idea.)

In the approach followed in this paper conditions on a general measure-valued path ϱ in order for this equation (2) to make sense will be formulated. Note however if $\varrho_t(\mathrm{d}b) \equiv \varrho(\mathrm{d}b)$ and the latter measure charges some polar set, infinities can occur in the reaction term of this equation. This leads to restrictions that must be placed on ϱ which will be made precise in the probabilistic development below.

There is a close relation between a subclass of quasilinear reaction diffusion equations (with regular reaction term) and both branching particle systems and superprocesses. For instance, Dynkin and Kuznetsov [DK95] and Le Gall [LG95] have exploited this relation with superprocesses to study the question of removable singularities for these equations. Gorostiza and Wakolbinger [GW95] have used both branching particle system and superprocess representations to obtain results on the long time behavior of systems of reaction diffusion equations.

In the same way, there is a relation between catalytic reaction diffusion equations and catalytic branching particle systems respectively superprocesses, and this may also lead to a probabilistic approach to the analysis of these equations. In addition the branching particle system viewpoint naturally leads to a "microscopic" level interpretation of the catalytic reaction as well as being of interest in its own right.

Catalytic branching has been studied already in a series of papers: [DFR91, DF91, DF94, Fle94, DF95, DFLM95, DFL95, FL95, DLM94, Dyn94a].

1.2 Intuitive description of catalytic branching

At the microscopic level we begin with a system of *reactant* particles and a spatially density field $\varrho = \{\varrho_t(b);\ t \geq 0,\ b \in \mathsf{R}^d\}$ of a *catalyst.* Here $\varrho_t(b)$ is understood as the generalized derivative $\varrho_t(\mathrm{d}b)/\mathrm{d}b$ at b of the (possibly singular) measure $\varrho_t(\mathrm{d}b)$. Let us assume that the reactant particles move independently in R^d according to standard Brownian motions W, except that each particle located at time t at b may die or branch with offspring generating function G at rate proportional to the "amount of catalyst $\varrho(t,b)$ present at time t at b".

Newly born particles start at the position of their parent but otherwise move independently.

Let $N(t)$ denote the (random) number of reactant particles at time t and $x_i(t)$ the (random) location of the ith particle at time t. Then $\sum_{i=1}^{N(t)} \delta_{x_i(t)}$ is the state of reactant at time t. If we start at time s with a single particle at a, this system of branching Brownian motions is described by its (transition) Laplace functional

$$v(s,t,a) \;:=\; P_{s,\delta_a} \exp -\textstyle\sum_{i=1}^{N(t)} \varphi\big(x_i(t)\big)$$

which satisfies the catalytic reaction diffusion equation

$$-\,\frac{\partial v(s,t,a)}{\partial s} \;=\; \frac{1}{2}\Delta v(s,t,a) + \varrho_s\Big(G\big(v(s,t,a)\big) - v(s,t,a)\Big), \qquad v\Big|_{s=t} = \mathrm{e}^{-\varphi},$$

where φ is a non-negative measurable function on R^d.

Heuristically, this equation can be reformulated (using the approach of Dynkin [Dyn94b]) as

$$\begin{aligned} v(s,t,a) \;&=\; \Pi_{s,a}\bigg[\exp\big[-L(s,t)\big]\exp\big[-\varphi(W_t)\big]\Big] \\ &\qquad\qquad + \int_s^t L(\mathrm{d}r)\exp\big[-L(s,r)\big]G\big(v(r,t,W_r)\big)\bigg] \end{aligned}$$

where $\Pi_{s,a}$ denotes the law of Brownian motion W starting at time s at a. Moreover, $L = L[W,\varrho]$ is a continuous additive functional of W, the so-called *collision local time* between a Brownian particle with path W and the catalytic medium ϱ which heuristically is given by

$$L[W,\varrho](s,t) = \int_s^t \mathrm{d}r \int \varrho_r(\mathrm{d}b)\,\delta_b(W_r).$$

Now $\delta_b(W_r)L(\mathrm{d}r)$ gives a more precise meaning to "the amount of catalyst $\varrho(r,b)$ present at time r at b" meat by a reactant particle with path W (when ϱ_r is a singular measure). This captures the microscopic view that a tagged Brownian particle with path W branches according to a "clock" given by the additive functional $L[W,\varrho]$.

Formally this covers the interesting case in which the catalyst also consists of diffusing particles: $\varrho_t = \sum_i \delta_{\gamma_i(t)}$. Then

$$L[W,\varrho](s,t) = \sum_i \int_s^t \mathrm{d}r\;\delta_{\gamma_i(r)}(W_r),$$

which makes sense in dimension $d = 1$.

If we assume that the offspring distribution of reactant has a finite second moment we obtain in the usual high density limit the *catalytic super-Brownian motion* $X = X^{\varrho} = \{X_t^{\varrho};\ t \geq 0\}$ in R^d, described by the log-Laplace function

$$v(s,t,a) \ = \ -\log P_{s,\delta_a} \exp - \int X_t^{\varrho}(\mathrm{d}b)\, \varphi(b)$$

which solves a special case of the catalytic reaction diffusion equation introduced in (1), namely

$$-\frac{\partial v(s,t,a)}{\partial s} \ = \ \frac{1}{2}\Delta v(s,t,a) - \varrho_s(a)\, v^2(s,t,a), \quad s \leq t, \qquad v\Big|_{s=t} = \varphi,$$

(recall $\varrho_s(a)$ is understood as the generalized density function of the measure $\varrho_s(\mathrm{d}a)$).

In the case in which $\varrho_t(\mathrm{d}b) \equiv \gamma\, \mathrm{d}b$, where γ is a (strictly) positive constant, X^{ϱ} is the usual (critical) *continuous super-Brownian motion (SBM).* However in applications the catalytic mass ϱ can be a singular measure $\varrho(\mathrm{d}b)$ (e.g. concentrated on a hypersurface), may vary in time, $\varrho_t(\mathrm{d}b)$, (*varying medium*), or even be sampled from a random object (*random medium*). For example, we could consider the situation in which ϱ is an ensemble of catalytic particles which also undergo branching with constant branching rate.

In fact, our *first objective* in this paper is to initiate the study of a catalytic branching model X^{ϱ} in which the catalytic mass process ϱ evolves itself as a super-Brownian motion with constant branching rate γ. Note that in dimensions $d \geq 2$ the collision local time of a pair of Brownian particles is always zero (recall that independent Brownian particles do not meet in $d \geq 2$.). Nevertheless we will see that in dimensions $d = 2, 3$ a Brownian particle does have a nontrivial collision local time with a super-Brownian catalytic medium ϱ, and that in these dimensions we can define a super-Brownian motion X^{ϱ} with a catalyst ϱ which is a super-Brownian motion (with constant branching rate γ).

The *second main objective* is to start the study of the qualitative behavior of this system. For example does the reacting species X_t^{ϱ} die out in the long time limit $t \to \infty$? In order to address this problem it is necessary to consider spatially homogeneous initial conditions ϱ_0 (e.g. Lebesgue measure ℓ), and for this reason in our basic construction we consider a class of infinite measures.

These models with *"one-way interaction"* can also be viewed as an intermediate step to spatial branching models with proper interaction. A model of "two-way killing" using collision local time is developed in Evans and Perkins [EP94], and a model with "mutually catalytic branching" will be studied in Dawson, Mueller and Perkins [DMP95].

1.3 Further details on the model to be constructed

Let us now discuss in more detail our case in which ϱ is actually sampled itself from a continuous SBM in R^d with a constant branching rate $\gamma > 0$. Consequently, ϱ serves as a *catalytic random medium* for $X = X^{\varrho}$. For simplicity, we call ϱ the *catalyst process* and X^{ϱ} the *catalytic SBM.*

In the particle picture this means in particular that an X-particle (*reactant*) may branch only if it is in the vicinity of a ϱ-particle (*catalyst*). In other words, branching of a reactant is controlled by the collision local time $L = L[W, \varrho]$ of its Brownian path W with all the Brownian paths of the catalyst (occupation density of this X-particle on all the ϱ-particles).

To be more specific, consider first the *one-dimensional* case $d = 1$. It is well-known that here the continuous SBM ϱ lives in the set of absolutely continuous measures. Therefore in this case the Radon-Nikodym densities $\varrho_t(b)$ of $\varrho_t(\mathrm{d}b)/\mathrm{d}b$ (with respect to the Lebesgue measure $\mathrm{d}b$) taken at b exist for each $t > 0$, even as a jointly continuous field $\{\varrho_t(b);\ t > 0,\ b \in \mathsf{R}\}$ (Konno and Shiga [KS88], Reimers [Rei89]). Thus, given ϱ,

$$L(\mathrm{d}r) := \varrho_r(W_r)\,\mathrm{d}r = \mathrm{d}r \int \varrho_r(\mathrm{d}b)\,\delta_b(W_r) \tag{3}$$

defines a continuous additive functional $L = L[W, \varrho]$ of Brownian motion W which we call the *Brownian collision local time (BCLT) of* ϱ. This L is used to govern the branching of a reactant with path W, in the given medium ϱ. At least intuitively, this makes clear that the *one*-dimensional X^{ϱ} as informally described above exists.

The situation changes dramatically for dimensions $d \geq 2$, since then the random measures $\varrho_t(\mathrm{d}b)$ are *singular* (Dawson and Hochberg [DH79]). Hence, one cannot use (3) to give an immediate definition for a BCLT of ϱ. Nevertheless, also in dimensions $d = 2$ and 3 Brownian collision local times $L = L[W, \varrho]$ exist non-trivially. For the case of a *finite* measure-valued SBM ϱ, see Evans and Perkins [EP94, combine Theorem 4.1 and Proposition 4.7]. As a consequence a non-degenerate catalytic SBM X^{ϱ} can be constructed in these dimensions, even in the infinite measure case.

For dimensions $d \geq 4$ however, the Brownian collision local time $L = L[W, \varrho]$ of ϱ degenerates to 0, since the closure of the graph of ϱ does not intersect with the graph of W, see Barlow and Perkins [BP93, Proposition 1.3]. In other words, here the reactants do not "feel" the catalyst, thus cannot branch. Therefore in these higher dimensions, if X^{ϱ} exists it must *degenerate* to the heat flow.

1.4 Sketch of main results

The *first objective* of the present paper is a rigorous construction of the continuous catalytic SBM X^{ϱ}, for dimensions $d \leq 3$. For this we assume as a rule that both the catalyst process ϱ and the catalytic SBM X^{ϱ} start off with a Lebesgue measure ℓ. A basic step is to establish the existence of the Brownian collision local time $L = L[W, \varrho]$ of ϱ for the present *infinite* measure-valued ϱ (Theorem 40 at p. 40). This heavily relies on results of Evans and Perkins [EP94]. Based on this step we then rigorously define X^ϱ (in dimensions $d \leq 3$) as a continuous process. Here in constructing the process X^ϱ, first ϱ is sampled and then the Markov process X^ϱ evolves in this chosen fixed medium ϱ (*quenched* approach).

Before we turn to the long-term behavior, it might be convenient at this stage to briefly recall some well-known facts on the (critical) continuous SBM ϱ with *constant* branching rate $\gamma > 0$. (For more details we refer to Dawson [Daw93, Chapter 4].)

If ϱ starts with a finite measure ϱ_0, then the total mass $\varrho_t(\mathsf{R}^d)$ degenerates to 0 after an a.s. finite time regardless of the dimension. When considering the total mass, the space structure becomes irrelevant since the branching rate is constant, resulting in Feller's critical branching diffusion.

Concerning the long-term behavior of ϱ starting with a Lebesgue measure ℓ, it is well-known that ϱ_t suffers local extinction as $t \to \infty$ almost surely, provided that $d = 1$, and stochastically if $d = 2$, whereas in all other dimensions convergence in law to a non-trivial steady state ϱ_∞ with expectation ℓ takes place (Dawson [Daw77]).

Our *second objective* is to initiate the study of the *long-term behavior* of X^ϱ, namely in the *one-dimensional* situation. Perhaps surprisingly at first sight, the picture is somewhat different from the results on one-dimensional branching models previously dealt with. In fact, for almost all realizations ϱ of the catalytic medium (starting with $\varrho_0 = \ell$), the random X_t^ϱ converges (stochastically) as $t \to \infty$ to the starting Lebesgue measure $X_0 = \ell$ (Theorem 48 at p. 49). Consequently, here we have *persistence,* that is, no loss of intensity in the limit (which in usual spatial branching models occurs only in higher dimensions). Moreover, in the finite measure case, the total mass process of X^ϱ converges a.s. to a limit which is *non*-deterministic and again with *full* expectation (Theorem 47 at p. 46).

On an intuitive level, this new type of long-term behavior of X^ϱ can be understood taking into account the *clumping* features of the (one-dimensional) catalyst process ϱ: At a late time, the spatially homogeneous ϱ_t has already died out in most regions of the space (see Dawson and Fleischmann [DF88]). Hence the reactant has only a small chance to meet the huge but rare remaining clumps of catalyst. Therefore the random medium ϱ effects a tagged X-particle only during some finite initial time interval. As a consequence, each finite initial mass X_0 undergoes a critical branching in a varying medium ϱ for some finite time only, but does not change after this, except for its spatial dispersion by the heat flow, which is irrelevant for the total mass process. In particular, it cannot lose any expectation of mass as $t \to \infty$. In the infinite measure case, additionally a law of large number effect has to be taken into account for all independently evolving finite parts of X_0. In simple terms, in dimension one, after a starting period, the random medium ϱ no longer influences the X-process, which then evolves (locally) as the heat flow.

This relatively complete picture concerning $d = 1$ however does not give a hint for the long-term behavior of X^ϱ in the more delicate dimensions $d = 2, 3$ which we leave open for a later study.

1.5 Outline

In the next section we prepare some *tools:* branching rate functionals K and related cumulant equation settings. Section 3 is devoted to the *construction* of the infinite measure-valued SBM X^K with general branching rate functional K, and the related occupation measure process Y^K. This construction is carried out by extending the construction of superprocesses to the case of *infinite* initial measures and also by introducing a localized version of Dynkin's admissibility condition [Dyn94b]. We give sufficient conditions on K for the existence of a *continuous* version of X^K, and also for the *absolute continuity* of the states of X^K and Y^K.

In Section 4 we compile facts on the SBM with constant branching rate we later need for our *catalyst process* ϱ. In particular, we extend Sugitani's [Sug89] joint continuity of the occupation density field to joint *Hölder* continuity, and in fact in a self-contained way. Section 5 contains our results on the *Brownian collision local time* $L[W, \varrho]$ of the catalyst process ϱ. These results extend the BCLT introduced by Barlow, Evans and Perkins [BEP91] to the case in which the super-Brownian motion has infinite initial measure. After these comprehensive preparations, the *existence* of the catalytic SBM X^ϱ is established in § 5.4. Finally, in Section 6 we study the *longtime* behavior of X_t^ϱ as $t \to \infty$ in dimension $d = 1$.

2 Preparation: Cumulant equation

The purpose of this section is to provide some equation tools needed in the next section to establish (Proposition 11 at p. 17 below) the existence of an *infinite* measure-valued SBM X^K with a given branching rate functional K, and, under an additional assumption on K, of a *continuous* version of X^K.

2.1 Preliminaries: Spaces

If E is a topological space, *measurability* always refers to the σ-field of all Borel subsets of E. Let $\mathcal{B}(E)$ denote the space of all measurable (real-valued) functions on E. We write $b\mathcal{B}(E)$ for the subspace of all *bounded* functions. A lower index $_+$ on a set refers to the subset of all of its non-negative members.

Usually we consider the Euclidean space $E = \mathsf{R}^d$ of dimension $d \geq 1$, and in this case we mostly omit the "argument" E, and simply write $\mathcal{B}$, $b\mathcal{B}$. Often also the case $E = I \times \mathsf{R}^d$ appears, where I is always a finite closed subinterval $[L, T]$, $L \leq T$, of R_+.

Fix a constant $p > d$ (with d the dimension of the phase space R^d), and a constant $\beta \geq 0$. Introduce the *reference function* ϕ_p defined by

$$\phi_p(a) := \left(1 + \beta |a|^2\right)^{-p/2}, \qquad a \in \mathsf{R}^d. \tag{4}$$

Let $\mathcal{B}^p$ denote the set of all those $\varphi \in \mathcal{B}$ satisfying $|\varphi| \leq c_\varphi \phi_p$ for some constant c_φ. If $\beta > 0$ then these are functions in $b\mathcal{B}$ with *decay* at least of order $|a|^{-p}$ as $a \to \infty$. For simplicity we then say that φ is of *p-potential decay.*

Let $\mathcal{B}^{p,I}$ denote the set of all those $\psi \in \mathcal{B}(I \times \mathsf{R}^d)$ which are *dominated* in the sense that $|\psi(s,\cdot)| \leq c_\psi \phi_p$, $s \in I$, for some constant c_ψ.

Write $\mathcal{C}$, $b\mathcal{C}$, $\mathcal{C}^p$, $\mathcal{C}^{p,I}$ for the subsets of all *continuous* functions in $\mathcal{B}$, $b\mathcal{B}$, $\mathcal{B}^p$, $\mathcal{B}^{p,I}$, respectively. $\mathcal{B}^p$ and $\mathcal{C}^p$ equipped with the norm $\|\varphi\| := \|\varphi/\phi_p\|_\infty$, $\varphi \in \mathcal{B}^p$ or $\mathcal{C}^p$, respectively, are Banach spaces. (Here $\|\cdot\|_\infty$ always denotes the supremum norm.) Similarly, $\mathcal{B}^{p,I}$ and $\mathcal{C}^{p,I}$, endowed with the norm

$$\|\psi\|_I := \sup\nolimits_{s\in I} \|\psi(s,\cdot)\|, \qquad \psi \in \mathcal{B}^{p,I} \text{ or } \mathcal{C}^{p,I}, \tag{5}$$

respectively, are Banach spaces. The subspace $\mathcal{C}^{p;\ell}$ of $\mathcal{C}^p$ of those functions φ which have a finite limit $\lim_{|b|\to\infty} \varphi(b)/\phi_p(b)$ is even *separable.* The same is true for the analogously defined $\mathcal{C}^{p,I;\ell}$.

We introduce the *"dual"* set $\mathcal{M}_p$ of all (locally finite non-negative) measures μ on R^d such that

$$\|\mu\|_p := \langle \mu, \phi_p \rangle < +\infty$$

where we set $\langle \mu, \varphi \rangle := \int \mu(\mathrm{d}b)\, \varphi(b)$. We endow $\mathcal{M}_p$ with the weakest topology such that the maps $\mu \mapsto \langle \mu, \varphi \rangle$ are continuous, for all $\varphi \in \mathcal{C}^{p;\ell}$. If $\beta > 0$, then $\mathcal{M}_p$ is the set of *p-tempered measures*, equipped with the so-called *p-vague* topology (note that $\mathcal{M}_p$ and its topology are independent of the choice of the constant $\beta > 0$). Otherwise (if $\beta = 0$), the class $\mathcal{M}_p$ degenerates to the set $\mathcal{M}_f$ of all *finite* measures on R^d endowed with the *weak* topology.

Denote by ψ_p the function on $I \times \mathsf{R}^d$ which equals ϕ_p constantly in time, that is $\psi_p(r,\cdot) \equiv \phi_p$. In analogy to $\mathcal{M}_p$ introduce the set $\mathcal{M}_p^I$ of all measures ν on $I \times \mathsf{R}^d$ such that $\langle \nu, \psi_p \rangle_I < +\infty$ where

$$\langle \nu, \psi \rangle_I := \int_{I\times \mathsf{R}^d} \nu(\mathrm{d}[r,b])\, \psi(r,b), \qquad \psi \in \mathcal{B}^{p,I}. \tag{6}$$

We furnish $\mathcal{M}_p^I$ with the weakest topology such that the maps $\nu \mapsto \langle \nu, \psi \rangle_I$ are continuous for all $\psi \in \mathcal{C}^{p,I;\ell}$.

Set $\|\mu\|_p := \langle \mu, \phi_p \rangle$, $\mu \in \mathcal{M}_p$, and write $\|\mu\|$ for the *total mass* $\mu(\mathsf{R}^d)$ if $\mu \in \mathcal{M}_f$.

The *open ball* in R^d with center a and radius r is denoted by $B(a,r)$.

2.2 Branching rate functional K

An essential ingredient for the branching models under consideration is the notion of a branching rate functional K which we will introduce this in this subsection.

Let $W = [W,\ \Pi_{s,a},\ s \geq 0,\ a \in \mathsf{R}^d]$ denote a *standard Brownian motion* in R^d, on canonical path spaces of continuous functions. (Although W is a time-homogeneous Markov process, for convenience we use the inhomogeneous setting $\Pi_{s,a}$.) Write p for its continuous *transition density* function,

$$\mathrm{p}(t,a,b) = \mathrm{p}(t,b-a) := (2\pi t)^{-d/2} \exp -\tfrac{|b-a|^2}{2t}\,, \qquad t>0, \quad a,b \in \mathsf{R}^d,$$

and $S = \{S_t\,;\ t \geq 0\}$ for the related semigroup. Set formally $S_t = 0$ for $t < 0$. Put

$$\Pi_{s,\mu} := \int \mu(da)\, \Pi_{s,a}\,, \qquad s \geq 0, \quad \mu \in \mathcal{M}_p\,,$$

for the "law" of W starting at time s in the point a "distributed" according to the (infinite) measure μ.

Recall that a (non-negative) functional $A = A[W]$ of W is called *additive* if, given W, it is a measure $A(\mathrm{d}r)$ on $\mathsf{R}_{++} := (0,+\infty)$, finite on bounded subintervals, and moreover, if $AJ := A(J)$ is measurable with respect to the universal completion of the σ-field generated by $\{W_r\,;\ r \in J\}$, for every open interval $J := (s,t)$ where $0 \leq s < t$.

Definition 1 (branching rate functional K) An additive functional $K = K[W]$ of the Brownian motion W is called a *branching rate functional*, if

(a) it is *continuous,* i.e. $K(\mathrm{d}r)$ does not carry mass at any single point set,

(b) it has *locally finite characteristic*, i.e. has the following finite expectations

$$\Pi_{s,a} \int_s^t K(\mathrm{d}r)\ \phi_p(W_r) < \infty, \qquad 0 \leq s < t, \quad a \in \mathsf{R}^d,$$

(c) and finally if it is *locally admissible*, i.e.

$$\sup_{a \in \mathsf{R}^d} \Pi_{s,a} \int_s^t K(\mathrm{d}r)\ \phi_p(W_r) \xrightarrow[{[s,t]\downarrow\{r_0\}}]{} 0, \qquad r_0 \geq 0. \tag{7}$$

Denote by $\mathbf{K}$ the set of all branching rate functionals, and by $\mathbf{K}_0$ the subset of those $K \in \mathbf{K}$ satisfying both (b) and (c) even with the reference function ϕ_p of (4) replaced by the constant function 1 (or setting $\beta = 0$ in (4)).

Write $K_n \uparrow K_0$ if $K_0, K_1, \ldots \in \mathbf{K}$ and if with probability one $K_n(J) \uparrow K_0(J)$ as $n \to \infty$ for all open intervals J of R_{++}. Call $K_1, K_2, \ldots \in \mathbf{K}_0$ an *approximating sequence* of $K \in \mathbf{K}$ if $K_n \uparrow K$. $\diamond$

Remark 2 (localization) (i) Dynkin [Dyn94b, § 3.3.3] called the functionals in $\mathbf{K}_0$ *admissible*. Our localized version of this (that is the definition of the set $\mathbf{K}$) is motivated by our aim later to cover the case of the collision local time $K = L = L[W, \varrho]$ of Brownian motion W with the continuous SBM ϱ starting with an (infinite) Lebesgue measure ℓ.

(ii) $\mathbf{K}_0$ is *dense* in $\mathbf{K}$ in the following sense: To each $K \in \mathbf{K}$ there exist an *approximating sequence* $K_n \uparrow K$. In fact,

$$K_n := \int_{(\cdot)} K(\mathrm{d}r)\ (1 \wedge n\phi_p)(W_r) \in \mathbf{K}_0\,, \qquad n \geq 1. \tag{8}$$

can be taken.

(iii) If $K(\mathrm{d}r)$ belongs to $\mathbf{K}$, then $\phi_p(W_r)K(\mathrm{d}r)$ belongs to $\mathbf{K}_0$. ◇

In § 3.1 below, such branching rate functionals K will be used to govern the branching of particles hidden in clouds of populations.

In order to conclude later for the existence of SBMs X^K with *finite higher moments* or even having *continuous paths*, we will need some additional conditions on K (which will be used already in § 2.4 below).

Definition 3 Let K be a branching rate functional, that is $K \in \mathbf{K}$.

(a) **(functionals in $\mathbf{K}^*$)** We say that K belongs to $\mathbf{K}^*$ if for each (finite) $I = [L, T] \subset \mathsf{R}_+$ there exists a constant κ_I such that

$$\sup_{s \in I} \Pi_{s,a} \int_s^T K(\mathrm{d}r)\ \phi_p^2(W_r) \ \leq\ \kappa_I\ \phi_p(a), \qquad a \in \mathsf{R}^d.$$

(That is, this supremum expression belongs to $\mathcal{B}_+^p$.)

(b) **(functionals in $\mathbf{K}^\xi$)** Let $\xi > 0$ be fixed. K is counted to belong to $\mathbf{K}^\xi$ if for each $N > 0$ there is a constant $c_N > 0$ such that

$$\Pi_{s,a} \int_s^t K(\mathrm{d}r)\ \phi_p^2(W_r) \ \leq\ c_N\ |t - s|^\xi\ \phi_p(a), \qquad 0 \leq s \leq t \leq N,\ \ a \in \mathsf{R}^d.$$

◇

Note that $\mathbf{K}^\xi \subseteq \mathbf{K}^*$. The set $\mathbf{K}^*$ is devoted for the existence of X^K with finite higher moments, whereas $\bigcup_{\xi>0} \mathbf{K}^\xi$ is aimed to X^K having continuous paths.

Example 4 (special K) (i) In the *constant branching rate* case $K(\mathrm{d}r) \equiv \gamma\,\mathrm{d}r$, this functional K is non-random and homogeneous in time and space and belongs to $\mathbf{K}_0 \cap \mathbf{K}^\xi$ with $\xi = 1$, and leads to the catalyst process ϱ; see Section 4 below.

(ii) In the *single point catalytic model* of [DF94] (where $d = 1$), the branching rate functional $K(\mathrm{d}r)$ is given by the Brownian local time $L^c(\mathrm{d}r)$ at a fixed point $c \in \mathsf{R}$, hence it also belongs to $\mathbf{K}_0 \cap \mathbf{K}^\xi$ but now with $\xi = 1/2$. ◇

2.3 Basic equation setting

A basic tool for super-Brownian motions is the so-called *cumulant equation* we now deal with. Fix a closed interval $I := [L,T],\ 0 \leq L \leq T$.

Proposition 5 (cumulant equation) *Fix a branching rate functional $K = K[W]$ in $\mathbf{K}$ and an additive functional $A = A[W]$ of Brownian motion W having a locally bounded characteristic, i.e.*

$$\sup\nolimits_{s\in I} \Pi_{s,a} A[s,T] \leq c_A \phi_p(a), \qquad a \in \mathsf{R}^d, \tag{9}$$

for some constant c_A. Then the following statements hold:

(a) (unique existence) *There is exactly one function $v =: v^I[A,K] \in \mathcal{B}_+^{p,I}$ satisfying the integral equation*

$$v(s,a) = \Pi_{s,a}\biggl[A[s,T] - \int_s^T K(\mathrm{d}r)\, v^2(r,W_r)\biggr], \qquad s\in I, \quad a\in\mathsf{R}^d. \tag{10}$$

(b) (continuity in K) *If $K_n \uparrow K_0$ in $\mathbf{K}$, the pointwise monotone convergence $v_n \downarrow v_0$ as $n\to\infty$ holds for the corresponding solutions $v_n := v^I[A,K_n]$ of equation* (10).

Example 6 (special A) An important special case of the additive functional A is given by $A(\mathrm{d}r) = \psi(r,W_r)\,\alpha(\mathrm{d}r)$ with $\psi \in \mathcal{B}_+^{p,I}$ and where α is a (non-random) finite measure on I. In particular, if $\alpha = \delta_T + \mathbf{1}_I(r)\,\mathrm{d}r$. In fact, the domination property (9) then follows from a corresponding property of the heat solution $S_{T-s}\varphi(a) = \Pi_{s,a}\varphi(W_T)$ with terminal condition $\varphi \in \mathcal{B}^p$ (see e.g. Lemma 4.1 in [DF88]):

$$\sup\nolimits_{s\in I} S_{T-s}\varphi\,(a) \leq \text{const}\ \|\varphi\|\,\phi_p(a) \tag{11}$$

(where the constant *const* depends on I). $\diamond$

Proof 1° (*localization*) If in (10) we replace K by K_n from (8), then by Theorem 3.4.2 of Dynkin [Dyn94b] there exists a unique non-negative bounded solution v_n of (10) (the non-negativity follows from the construction given there). Then from (10) and (9) we get

$$0 \leq v_n(s,a) \leq \Pi_{s,a} A[s,T] \leq c_A\phi_p(a), \qquad s\in I, \quad a\in\mathsf{R}^d. \tag{12}$$

Hence, the solution v_n is dominated. Moreover, by construction, v_n is monotone non-increasing in n. Denote by v the pointwise limit of v_n as $n\to\infty$.

2° (*existence*) The limit $v \geq 0$ is obviously dominated. It will solve (10) if we show that

$$\Pi_{s,a}\int_s^T K_n(\mathrm{d}r)\, v_n^2(r,W_r) \underset{n\to\infty}{\longrightarrow} \Pi_{s,a}\int_s^T K(\mathrm{d}r)\, v^2(r,W_r) \tag{13}$$

for each $s \in I$ and $a \in \mathsf{R}^d$. This will be done via *two-sided estimates.*

The l.h.s. can be estimated below by switching from v_n to v. Then the r.h.s. appears as the finite monotone limit as $n \to \infty$, since by (12) the l.h.s. is dominated, uniformly in n.

Concerning the other direction, replace K_n by K at the l.h.s. of (13). Assuming for the moment that we still have a finite expression, we will be ready, again by monotone convergence.

To show the mentioned finiteness, by (12) it suffices to show that

$$\Pi_{s,a} \int_s^T K(\mathrm{d}r)\ \phi_p(W_r) < \infty, \qquad s \in I, \quad a \in \mathsf{R}^d.$$

But this follows from the local admissibility (7) by taking a sufficiently small partition of the interval (s,T), and using the Markov property.

Altogether, we showed that the limit v satisfies (10), giving the existence claim in (a).

$3°$ (*continuity*) Assuming only $K_n \uparrow K$ in $\mathbf{K}$ in the arguments of the previous step of proof, and uniqueness of solutions which will be shown in the next step, the continuity statement (b) follows by the same two-sided estimation arguments.

$4°$ (*uniqueness*) Assume for the moment that v_1 and v_2 are *different* solutions to (10). Let T_0 denote the supremum over all $s \in I$ such that $v_1(s) \neq v_2(s)$. If $T_0 = L$, then from (10) we get $v_1(L) - v_2(L) = 0$ yielding a contradiction. On the other hand, if $T_0 > L$, then by construction and using the domination property (12),

$$|v_1 - v_2|(s,a) \ \le\ c\,\|v_1 - v_2\|_s\, \Pi_{s,a} \int_s^{T_0} K(\mathrm{d}r)\ \phi_p(W_r) \tag{14}$$

for $s \in [L, T_0)$ and $a \in \mathsf{R}^d$, and some constant $c > 0$, where (in this proof) $\|\cdot\|_s$ refers to the supremum norm on $[s, T_0) \times \mathsf{R}^d$. Choose $\varepsilon > 0$ such that $c\varepsilon < 1$, and by the local admissibility (7) take $L_0 \in [L, T_0)$ such that

$$\Pi_{s,a} \int_s^{T_0} K(\mathrm{d}r)\ \phi_p(W_r) \ \le\ \varepsilon, \qquad L_0 \le s < T_0\,, \quad a \in \mathsf{R}^d.$$

Then from (14) we get

$$|v_1 - v_2|(s,a) \ \le\ c\,\varepsilon\, \|v_1 - v_2\|_{L_0}\,, \qquad s \in [L_0\,, T_0), \quad a \in \mathsf{R}^d,$$

which yields a contradiction. This establishes uniqueness and finishes the proof of the proposition. ■

The setting of the cumulant equation, provided in Proposition 5, will enable us later to *construct* a SBM X^K with branching rate functional K, see § 3.1.

2.4 Derivatives of solutions to a small parameter

In order to deal later with higher moment formulas of the processes to be constructed, in this subsection we will derive some estimates for *higher derivatives* of solutions of the cumulant equation with respect to a small parameter θ. For typographical simplification, we introduce the following convention.

Convention 7 (derivatives) If a function f on some space additionally depends on a parameter $\theta \geq 0$, write $\mathsf{D}^n f$ for the nth partial derivative of f with respect to θ (if exists), and $f^{(n)}$ for $\mathsf{D}^n f\big|_{\theta=0+}$. ◇

In this subsection we will actually work with the following hypothesis. Recall that $I = [L,T]$, $0 \leq L \leq T$.

Hypothesis 8 Fix $K \in \mathbf{K}^*$ (recall Definition 3 (a)), $f \in \mathcal{B}_+^{p,I}$, and let $\theta \geq 0$. Assume that $v = v_\theta \in \mathcal{B}_+^{p,I}$ solves the equation

$$v(s,a) \;=\; \theta\, f(s,a) - \Pi_{s,a} \int_s^T K(\mathrm{d}r)\; v^2(r, W_r), \qquad s \in I, \quad a \in \mathsf{R}^d, \tag{15}$$

and set

$$u \;=\; u_\theta := \;\theta\, f - v. \tag{16}$$

◇

Start with the following simple lemma (recall the Convention 7):

Lemma 9 (recurrence schema for derivatives) *Under Hypothesis* 8 *we have*

$$u^{(1)} = f - v^{(1)} \;=0, \tag{17}$$

whereas the sequence $\{u^{(k)};\ k \geq 2\}$ *of functions on* $I \times \mathsf{R}^d$ *is uniquely determined by the following recurrence schema:*

$$\left.\begin{aligned} &u^{(2)}(s,a) = 2\,\Pi_{s,a} \textstyle\int_s^T K(\mathrm{d}r)\, f^2(r, W_r),\\ &u^{(k)}(s,a) = -2\,\Pi_{s,a}\ \textstyle\int_s^T K(\mathrm{d}r)\, k\, f(r, W_r)\, u^{(k-1)}(r, W_r)\\ &\qquad + \textstyle\sum_{2 \leq i \leq k-2}\ \binom{k}{i}\, \Pi_{s,a}\ \int_s^T K(\mathrm{d}r)\, \big[u^{(k-i)} u^{(i)}\big](r, W_r), \quad k \geq 3. \end{aligned}\right\} \tag{18}$$

Proof By definition,

$$u(s,a) = \Pi_{s,a} \int_s^T K(\mathrm{d}r)\, v^2(r, W_r), \qquad s \in I, \quad a \in \mathsf{R}^d.$$

Therefore,

$$u^{(n)}(s,a) = \Pi_{s,a} \int_s^T K(\mathrm{d}r) \Big[\textstyle\sum_{i=0}^n \binom{n}{i}\, v^{(n-i)} v^{(i)}\Big](r, W_r), \qquad n \geq 0. \tag{19}$$

Now (15) implies $v \leq \theta f$, hence $v^{(0)} = 0$, giving $u^{(0)} = 0 = u^{(1)}$. On the other hand, from the definition (16) we then conclude (17) and $v^{(k)} = -u^{(k)},\ k \geq 2$. Inserting into (19) implies (18). That all operations in this proof make sense can be seen by induction arguments as in the proof of the following lemma. ∎

In order to get later moment formulas also in the case of signed test functions, we now allow f to be *signed*. Recall the definition of the norms $\|\cdot\|$ and $\|\cdot\|_I$ in $\mathcal{B}^p$ and $\mathcal{B}^{p,I}$, respectively, introduced in § 2.1.

Lemma 10 (estimate for the solution of the recurrence scheme) *There are constants $c_k > 0,\ k \geq 2$, such that $\limsup_{k\to\infty} c_k^{1/k} < +\infty$ and that, for given $K \in \mathbf{K}^*$ and $f \in \mathcal{B}^{p,I}$, the solution $\{u^{(k)};\ k \geq 2\}$ of the recurrence schema* (18) *exists uniquely and satisfies*

$$\|u^{(k)}(s,\cdot)\| \leq k!\, c_k\, \|f\|_I^k\, \kappa_I^{k-1}, \qquad s \in I, \quad k \geq 2, \tag{20}$$

with κ_I from Definition 3 (a).

Proof Define $\{c_k\,;\ k \geq 1\}$ as the solution of the following recursive system:

$$c_1 = 1, \qquad c_k = C \textstyle\sum_{1 \leq i \leq k-1} c_{k-i}\, c_i\,, \quad k \geq 2, \tag{21}$$

with $C = 1$. Then the power series $g(\theta) := \sum_{k\geq 1} c_k\, \theta^k,\ \theta > 0$, satisfies the quadratic equation $g(\theta) - \theta = C\, g^2(\theta)$, which can be solved if and only if $\theta \leq (4C)^{-1}$. Hence, the radius of convergence of the power series g is positive, and the lim sup condition holds. With this sequence $c_1, c_2, \ldots$, it is very easy to verify (20) by induction. ∎

3 Super-Brownian motion X^K

The main purpose of this section is the construction of a *continuous* super-Brownian motion X^K with branching rate functional K in $\mathbf{K}^\xi$ (Theorem 17 at p. 21).

3.1 SBM X^K with branching rate functional K

The present paper is based on the model of an $\mathcal{M}_p$-valued (critical) *super-Brownian motion $X = X^K$ with branching rate functional $K \in \mathbf{K}$* we now introduce. Originally it goes back to Dynkin [Dyn91]. There X^K was constructed as an $\mathcal{M}_f$-valued Markov process under restricted conditions on K, which had been removed in [Dyn94b], except keeping the condition $K \in \mathbf{K}_0$.

Proposition 11 (SBM X^K) *Fix* $K \in \mathbf{K}$ *(recall Definition* 1*).*

(a) (existence) *There exists a (time-inhomogeneous) Markov process denoted by* $X = X^K = \left[X,\ P_{s,\mu}\,,\ s \geq 0,\ \mu \in \mathcal{M}_p\right]$ *with Laplace transition functional*

$$P_{s,\mu} \exp - \langle X_t\,, \varphi\rangle \;=\; \exp - \bigl\langle \mu, v(s,t,\cdot)\bigr\rangle, \tag{22}$$

$0 \leq s \leq t,\;\; \mu \in \mathcal{M}_p\,,\;\; \varphi \in \mathcal{B}^p_+$*, where* $v(\cdot,t,\cdot) \geq 0$ *is uniquely determined by the cumulant equation*

$$v(s,t,a) = \Pi_{s,a}\biggl[\varphi(W_t) - \int_s^t K(\mathrm{d}r)\; v^2(r,t,W_r)\biggr], \tag{23}$$

$0 \leq s \leq t,\;\; a \in \mathsf{R}^d$.

(b) (expectation and covariance formulas) *For* $0 \leq s \leq t, t_1, t_2$ *and* μ *in* $\mathcal{M}_p$ *as well as* $\varphi, \psi \in \mathcal{B}^p_+$,

$$P_{s,\mu}\,\langle X_t\,, \varphi\rangle \;=\; \Pi_{s,\mu}\varphi(W_t) \;=\; \langle \mu, S_{t-s}\varphi\rangle \;=\; \langle S_{t-s}\mu, \varphi\rangle \;<\; \infty,$$

$$\mathcal{C}\mathrm{ov}_{s,\mu}\bigl[\langle X_{t_1}, \varphi\rangle,\langle X_{t_2}, \psi\rangle\bigr] = 2\Pi_{s,\mu} \textstyle\int_s^{t_1\vee t_2} K(\mathrm{d}r)\, S_{t_1-r}\,\varphi\,(W_r)\, S_{t_2-r}\psi\,(W_r).$$

(c) (continuity in K) *If* $K_n \uparrow K$ *in* $\mathbf{K}$*, then* $X^{K_n} \to X^K$ *as* $n \to \infty$ *in the sense of the convergence of all finite-dimensional distributions.*

This X^K is called a (critical) *super-Brownian motion (SBM) with branching rate functional* K. (In fact, a hidden Brownian particle at position W_r at time r branches with rate $K[W](\mathrm{d}r)$.) Note that the covariance in (b) could be infinite at this stage. This was the reason that we introduced the subset $\mathbf{K}^*$ of $\mathbf{K}$ which we will exploit in § 3.2.

Proof Fix for the moment $\varphi \in \mathcal{B}^p_+$, $t \geq 0$ and set $I = [0,t]$ as well as $A(\mathrm{d}r) = \varphi(W_t)\,\delta_t(\mathrm{d}r)$. As already mentioned in Example 6, this A satisfies (9). Hence, (23) is a special case of (10) if we set $v(\cdot,t,\cdot) := v^{[0,t]}[A,K]$.

The existence of $\mathcal{M}_f$-valued Markov processes X^{K_n} satisfying (a) and (b) with K replaced by the approximating $K_n \in \mathbf{K}_0$ from (8) is guaranteed by Dynkin [Dyn94b, Theorem 3.4.1]. The extension to $\mathcal{M}_p$-valued Markov processes X^{K_n} is possible by the domination property (12). Based on $K_n \uparrow K$ and the monotone convergence $v_n \downarrow v$ as in Proposition 5 (b), we conclude for the convergence of the corresponding Laplace functionals (22), for fixed s, μ, t. Note that the limiting Laplace functional is proper since $v \downarrow 0$ if $\varphi \downarrow 0$, recall (12). Hence, via (22) a limiting random measure X_t is determined. Moreover, by the semigroup structure of the solutions to (23) (which is based on the uniqueness of solutions) we may construct the laws of vectors $\bigl[X_{t_1}, \ldots, X_{t_k}\bigr]$. These compatible finite-dimensional distributions determine a Markov process X (which is independent of the choice of the approximating sequence K_n, $n \geq 1$). This gives the existence claim (a). The continuity statement (c) is obvious by monotone convergencies. The moment formulas (b) also follow by monotone convergence from known ones; see, for instance, Dynkin [Dyn91, formulas (1.28) and (1.30)]. This finishes the proof. ∎

3.2 SBM X^K with finite higher moments

The existence proof of a continuous version of X^K for $K \in \mathbf{K}^\xi$ will be based on Kolmogorov's method of moments. Indeed, despite that branching is governed by a fairly general functional K (for instance, think of the singular K in the single point-catalytic model, see Example 4 (ii)), the SBM X^K turns out to have (finite) *moments of all orders*, provided that $K \in \mathbf{K}^*$.

The following estimates will be provided for a *bivariate process* $[X, Y]$. (One can think of Y as the occupation measure related to X, that is $Y(\mathrm{d}r\,\mathrm{d}b) = \mathrm{d}r\, X_r(\mathrm{d}b)$; a justification will be given in § 3.7.) Recall the notation $\langle \nu, \psi \rangle_I$ from (6).

Hypothesis 12 (bivariate process) Let

$$[X,Y] \;=\; [X^K, Y^K] \;=\; \Big[[X,Y],\; P_{s,\mu},\; s \geq 0,\; \mu \in \mathcal{M}_p\Big]$$

be a (time-inhomogeneous) Markov process such that

$$P_{s,\mu} \exp\Big[-\langle X_t, \theta_1 \varphi\rangle - \langle Y_t, \theta_2 \psi\rangle_{[s,t]}\Big] = \exp -\big\langle \mu, v(s,t,\cdot)\big\rangle, \tag{24}$$

$0 \leq s \leq t,\;\; \mu \in \mathcal{M}_p\,,\;\; \varphi \in \mathcal{B}_+^p,\;\; \psi \in \mathcal{B}_+^{p,[0,t]},\;\; \theta_1, \theta_2 \geq 0$, where, for $\theta_1, \theta_2, \varphi, \psi, t$ fixed, $v(\cdot, t, \cdot)$ solves the equation

$$v(s,t,a) = f(s,a) - \Pi_{s,a} \int_s^t K(\mathrm{d}r)\; v^2(r,t,W_r), \tag{25}$$

$0 \leq s \leq t,\;\; a \in \mathsf{R}^d$, with

$$f := \theta_1 f_1 + \theta_2 f_2 \tag{26}$$

where $f_1, f_2 \in \mathcal{B}_+^{p,[0,t]}$, and $K \in \mathbf{K}^*$. ◇

Replacing f by $\theta f,\; \theta \geq 0$, and differentiating at $\theta = 0+$, by (17) we get the *expectation formula*

$$P_{s,\mu} \Big[\, \langle X_t\,, \theta_1 \varphi\rangle + \langle Y_t\,, \theta_2 \psi\rangle_{[s,t]} \,\Big] = \big\langle \mu\,, f(s,\cdot)\big\rangle. \tag{27}$$

Lemma 13 (recursive schema for centered moments) *Fix* $0 \leq s \leq t,\;\; \mu$ *in* $\mathcal{M}_p\,,\;\; \varphi \in \mathcal{B}_+^p,\;\; \psi \in \mathcal{B}_+^{p,[0,t]}$, *and* $\theta_1, \theta_2 \geq 0$. *Under Hypothesis* 12 *and with respect to* $P_{s,\mu}$, *the centered random variable*

$$Z = Z^K := \big\langle \mu\,, f(s,\cdot)\big\rangle - \langle X_t\,, \theta_1 \varphi\rangle - \langle Y_t\,, \theta_2 \psi\rangle_{[s,t]} \tag{28}$$

has moments of all orders. Moreover, if v *is taken from* (25) *but with* f *replaced by* $\theta f,\; \theta \geq 0$, *and if* u *is defined as in* (16), *then the moments of* Z *satisfy the recursive schema*

$$\left.\begin{aligned} P_{s,\mu}\, Z^k &= \big\langle \mu\,, u^{(k)}(s,t,\cdot)\big\rangle \\ &\quad + \textstyle\sum_{2 \leq j \leq k-2} \binom{k-1}{j} \big\langle \mu\,, u^{(k-j)}(s,t,\cdot)\big\rangle\, P_{s,\mu}\, Z^j\,, \qquad k \geq 2, \end{aligned}\right\} \tag{29}$$

with the $\{u^{(k)};\ k \geq 2\}$ *from* (18).

Proof From (24):

$$P_{s,\mu} \exp[\theta Z] = \exp\langle \mu, u(s,t,\cdot)\rangle.$$

Differentiating once with respect to θ yields (recall Convention 7)

$$P_{s,\mu} Z \exp[\theta Z] \;=\; \langle \mu, D^1 u(s,t,\cdot)\rangle \exp\langle \mu, u(s,t,\cdot)\rangle.$$

For $k \geq 2$, differentiate now $k-1$ times at $\theta = 0+$ to obtain

$$P_{s,\mu} Z^k = \sum_{j=0}^{k-1} \binom{k-1}{j} \langle \mu\,, u^{(k-j)}(s,t,\cdot)\rangle\, P_{s,\mu} Z^j$$

Because of

$$P_{s,\mu} Z \;\equiv\; 0 \;\equiv\; \langle \mu\,, u^{(1)}(s,t,\cdot)\rangle,$$

(recall (27)), the summands for $j = 1$ and $j = k-1$ disappear. Thus, (29) follows. That all moments exist finitely can be justified by induction arguments as in the proof of Lemma 15 below. ■

Remark 14 (moments in the case of signed test functions) Recalling (26), it is easy to see that the solutions of both recursive schemes (18) and (29) are polynomials in $[\theta_1\,, \theta_1] \geq 0$. This justifies to switch to *signed* test functions $\varphi \in \mathcal{B}^p$ and $\psi \in \mathcal{B}^{p,[0,t]}$ in both schemes, and Lemma 13 still *remains valid* for these φ and ψ. ◇

3.3 Some estimates for higher centered moments of X^K

To get some estimates of centered moments for increments of the SBM X^K in the case $K \in \mathbf{K}^\xi$, we will proceed as in our paper [DF94]. Start with the following result.

Lemma 15 (higher centered moment estimates of X^K) *Fix* $N > 0$, *and* $K \in \mathbf{K}^\xi$ *for some* $\xi > 0$. *Then to each* $k \geq 2$ *there exists a constant* c_k *such that for the centered moments of the SBM* X^K *the following estimates hold:*

$$\left|P_{s,\mu}\langle S_{t-s}\mu - X_t\,, \varphi\rangle^k\right| \;\leq\; c_k\ (t-s)^{k\xi/2}\ \|\varphi\|^k\ \sum_{i=1}^{k-1} \|\mu\|_p^i\,, \tag{30}$$

$0 \leq s \leq t \leq N,\ \ \mu \in \mathcal{M}_p\,,\ \ \varphi \in \mathcal{B}^p\,.$

Proof First assume $\varphi \in \mathcal{B}_+^p$ and notice that X^K fits into Hypothesis 12 by setting $\theta_2 = 0$ and $[X, Y] = [X^K, 0]$, and $f_1(s,a) = \Pi_{s,a}\varphi(W_t)$ (recall Remark 14 and Example 6). Set $\theta_1 = 1$ and note that by the domination (11),

$$\|f_1\|_{[0,t]} \leq \text{const}\,\|\varphi\| \tag{31}$$

(with $\|\cdot\|_I$ introduced in (5)). If now $k=2$, then the inequality (30) directly follows from (29) and (20). For a proof by induction, consider $k \geq 3$ and assume that (30) is true for the numbers 2, ..., $k-1$. Then from (29), (20), and (31) we get

$$\left|P_{s,\mu}\left\langle S_{t-s}\mu - X_t\,, \varphi\right\rangle^k\right| \leq \text{const}\left(\|\varphi\|^k\|\mu\|_p(t-s)^{(k-1)\xi}\right.$$
$$\left.\textstyle\sum_{2\leq j\leq k-2}\|\mu\|_p\,\|\varphi\|^{k-j}(t-s)^{(k-j-1)\xi}\|\varphi\|^j(t-s)^{j\xi/2}\sum_{i=1}^{j-1}\|\mu\|_p^i\right).$$

But

$$(k-1) \text{ and } (k-j-1)+j/2 \text{ are both bounded below by } k/2 \tag{32}$$

(for the considered j,k). Thus we can continue to estimate from above to arrive at the r.h.s. of (30). This completes the proof by induction. ■

3.4 Hölder continuous SBM X^K

The main purpose of this subsection is to introduce a *continuous* SBM X^K with branching rate functional $K \in \mathbf{K}^\xi$. As a preparation we need a further lemma.

Lemma 16 (estimates of centered moments of increments of X^K) *Fix $N>0$, $k \geq 1$, and $K \in \mathbf{K}^\xi$ for some $\xi > 0$. Then for the increments of the centered process*

$$Z_t := X_t^K - P_{s,\mu}X_t^K, \qquad t \geq s, \tag{33}$$

we get the following even moment estimates:

$$P_{s,\mu}\left\langle Z_{t+h} - Z_t\,, \varphi\right\rangle^{2k} \;\leq\; \text{const}\left[\|S_h\varphi-\varphi\|^{2k} + h^{k\xi}\|\varphi\|^{2k}\right]\textstyle\sum_{i=1}^{2k-1}\|\mu\|_p^i\,,$$

$0 \leq s \leq t \leq t+h \leq N$, $\mu \in \mathcal{M}_p\,,\ \varphi \in \mathcal{B}^p$.

Proof This follows from Lemma 15 along the lines of the Proof of Lemma 3.2.2 in [DF94], with the obvious modifications related to the present time-inhomogeneity and infinite measure case. ■

Let $\mathcal{D}_0 = \{\varphi_1\,, \varphi_2\,, ...\}$ denote a countable subset of the domain of definition of the "generator" $\Delta/2$ of the strongly continuous semigroup S acting on $\mathcal{C}^{p;\ell}$, which is a dense subset of the separable Banach space $\mathcal{C}^{p;\ell}$. We define a metric ρ_p on $\mathcal{M}_p$ by

$$\rho_p(\mu,\nu) := \sum_{m=1}^{\infty} 2^{-m}\left(1 \wedge \left|\langle\mu,\varphi_m\rangle - \langle\nu,\varphi_m\rangle\right|\right), \qquad \mu,\nu \in \mathcal{M}_p\,, \tag{34}$$

which just generates the topology in $\mathcal{M}_p$. Now we are ready to state the *main result* of this section:

Theorem 17 *Fix a branching rate functional $K \in \mathbf{K}^\xi$ for some $\xi > 0$.*

(a) (Hölder continuity of the centered process) *Fix $N > 0$, $\mu \in \mathcal{M}_p$, $k \geq 1$, and $\varepsilon \in (0, \xi/2)$. There is a modification $\widetilde{Z}$ of the centered process Z of (33) such that*

$$\sup_{0 \leq s \leq N} P_{s,\mu} \left[\sup_{s \leq t \leq t+h \leq N} \left| \left\langle \widetilde{Z}_{t+h} - \widetilde{Z}_t \, , \varphi \right\rangle \right| / h^\varepsilon \right]^k < +\infty, \quad \varphi \in \mathcal{D}_0. \quad (35)$$

In particular, $P_{s,\mu}$-almost surely, $\widetilde{Z}$ has locally Hölder continuous paths of order ε (in the metric ρ_p).

(b) (continuity of the SBM) *Since $K \in \mathbf{K}^\xi$, there is a modification $\widetilde{X}$ of the super-Brownian motion $X = X^K$ of Proposition 11 with continuous paths.*

Proof Fix ξ, K, ε, N and $\varphi \in \mathcal{D}_0$ as in the theorem. Then

$$\|S_h \varphi - \varphi\| \leq \text{const } h \|\Delta \varphi\| = \text{const } h, \qquad h \geq 0.$$

Therefore Lemma 16 implies that, for some constants c_k,

$$P_{s,\mu} \left| \left\langle Z_{t+h} - Z_t \, , \varphi \right\rangle \right|^{2k} \leq c_k \, h^{k\xi} \textstyle\sum_{i=1}^{2k-1} \|\mu\|_p^i \, , \quad (36)$$

$0 \leq s \leq t \leq t+h \leq N, \ k \geq 1.$

(a) (*Hölder continuity*) Since $\mathcal{D}_0$ is converging determining in $\mathcal{M}_p$, by Theorem 1.2.1 of Revuz and Yor [RY91] we conclude from (36) that there is a modification $\widetilde{Z}$ of Z such that

$$\sup\nolimits_{0 \leq s \leq N} P_{s,\mu} \left[\sup\nolimits_{s \leq t \leq t+h \leq N} \left| \left\langle \widetilde{Z}_{t+h} - \widetilde{Z}_t \, , \varphi \right\rangle \right| / h^\alpha \right]^{2k} < +\infty, \qquad \varphi \in \mathcal{D}_0,$$

for $\alpha \in \left(0, \frac{k\xi - 1}{2k}\right)$. For all k sufficiently large, we can set $\alpha = \varepsilon$ getting (35) for sufficiently large even k. But then (35) holds for all $k \geq 1$. Based on the definition (34) of the metric ρ_p, by [Daw93, Corollary 3.7.3] we get the claimed Hölder continuity.

(b) (*continuity*) Since $P_{s,\mu} X_t^K = S_{t-s}\mu$ and the map $t \mapsto S_t \mu$ is continuous, we may set $\widetilde{X}_t = \widetilde{Z}_t + S_{t-s}\mu, \ t \geq s$. This finishes the proof. ■

In the following, in the case $K \in \mathbf{K}^\xi$ we tacitly always work with a *continuous* modification according to Theorem 17 writing again X instead of $\widetilde{X}$. As an immediate consequence of Theorem 17 (a) we get:

Corollary 18 (Hölder continuous SBM) *Let $K \in \mathbf{K}^\xi$ for some $\xi > 0$ and $\mu \in \mathcal{M}_p$, $s \geq 0$. Then X^K is locally Hölder continuous of order $\varepsilon < \frac{\xi}{2}$ with $P_{s,\mu}$-probability one if and only if $t \mapsto S_{t-s}\mu$ is locally Hölder continuous of order ε.*

Remark 19 Hölder continuities of X^K had been considered in the constant branching rate case $K(\mathrm{d}r) = \gamma\,\mathrm{d}r$ by Reimers [Rei89], Dawson [Daw93, Corollary 3.7.3], and Schied [Sch95]. It seems for us that continuous versions of X^K had been established so far in special cases only, as $K(\mathrm{d}s) = f(W_s)\,\mathrm{d}s$ for $f \in b\mathcal{C}_+$, or $K(\mathrm{d}s) = \delta_c(W_s)\,\mathrm{d}s$. For the former case when branching occurs at site b with rate $f(b)$, see Konno and Shiga [KS88], whereas continuity for the latter finite measure-valued single point-catalytic model was proved in Dawson and Fleischmann [DF94]. $\diamond$

3.5 Convergence of the total mass process

As an immediate consequence of the constructions so far we get the following result:

Proposition 20 (convergence of the total mass process) *Assume that $K \in \mathbf{K}^\xi$ for some $\xi > 0$, and let $s \geq 0$ and $\mu \in \mathcal{M}_f$. Then*

$$\lim\nolimits_{t\to\infty} \|X_t^K\| \quad \textit{exists} \;\; P_{s,\mu}\text{–}a.s.$$

and has an expectation bounded by the total initial mass $\|\mu\|$.

Proof By Theorem 17(b), the total mass process $\{\|X_t^K\|;\ t \geq s\}$ is $P_{s,\mu}$-a.s. continuous (take β in the definition of the reference function ϕ_p). Actually, by the Markov property and the expectation formula in Proposition 11(b), it is a continuous non-negative *martingale*. Then the statement follows from a martingale convergence theorem (see, for instance, [RY91, Corollary 2.2.11]) and Fatou's lemma. ■

3.6 X^K with absolutely continuous states

In the following, we call $\underline{\varepsilon} = \{\varepsilon_n\,;\ n \geq 1\}$ a *zero-sequence* if $\varepsilon_n > 0,\ n \geq 1$, and $\varepsilon_n \to 0$ as $n \to \infty$.

Proposition 21 (absolutely continuous states of X^K) *Consider $K \in \mathbf{K}^*$, $0 \leq s < t$, and $\mu \in \mathcal{M}_p$. Assume that there is a Lebesgue zero set $N \subset \mathsf{R}^d$ with the following property. To each $z \in \mathsf{R}^d \backslash N$ there exists a zero sequence $\underline{\varepsilon} = \underline{\varepsilon}(z)$ such that*

$$\begin{aligned} &\Pi_{s,\mu} \textstyle\int_s^t K(\mathrm{d}r)\, \mathrm{p}^2\big(\varepsilon_n(z) + t - r, W_r\,, z\big) \\ &\qquad \xrightarrow[n\to\infty]{} \Pi_{s,\mu} \textstyle\int_s^t K(\mathrm{d}r)\, \mathrm{p}^2(t - r, W_r\,, z) \;<\; +\infty. \end{aligned} \tag{37}$$

Then the $L^2(P_{s,\mu})$-limit $x_t(z)$ of

$$x_t^n(z) := \big\langle X_t\,,\ \mathrm{p}\big(\varepsilon_n(z), \cdot, z\big)\big\rangle$$

exists as $n \to \infty$, *for each* $z \in \mathsf{R}^d \backslash N$. *Moreover, with respect to* $P_{s,\mu}$, *the random measure* $X_t(\mathrm{d}z)$ *is absolutely continuous:*

$$P_{s,\mu}\Big(X_t(\mathrm{d}z) = x_t(z)\,\mathrm{d}z\Big) = 1.$$

The following moment formulas hold:

$$\begin{aligned} P_{s,\mu}\, x_t(z) &= S_{t-s}\mu(z), \qquad z \in \mathsf{R}^d, \\ Var_{s,\mu}\, x_t(z) &= 2\,\Pi_{s,\mu} \int_s^t K(\mathrm{d}r)\, \mathrm{p}^2(t-r, W_r, z), \qquad z \in \mathsf{R}^d \backslash N. \end{aligned}$$

Example 22 (single point-catalytic model) If $d=1$, $K(\mathrm{d}r) = \delta_c(W_r)\,\mathrm{d}r$, and $N = \{c\}$, then (37) holds, and we recover a result of [DF91]. $\diamond$

Proof of Proposition 21 1° (*convergence of expectations*) First of all, by the expectation formula in Proposition 11 (b),

$$P_{s,\mu}\, x_t^n(z) = S_{\varepsilon_n + t - s}\mu(z) \underset{n\to\infty}{\longrightarrow} S_{t-s}\mu(z), \qquad z \in \mathsf{R}^d,$$

which is the desired limiting expectation.

2° (*Cauchy sequence*) We want to show that $x_t^n(z)$ is Cauchy in $L^2(P_{s,\mu})$ as $n \to \infty$, for each $z \in \mathsf{R}^d \backslash N$. By Step 1°, it suffices to show that

$$Var_{s,\mu}\big[x_t^n(z) - x_t^{n'}(z)\big] \underset{n,n'\to\infty}{\longrightarrow} 0.$$

But by the covariance formula in Proposition 11 (b), these variances equal to

$$= 2\,\Pi_{s,\mu} \int_s^t K(\mathrm{d}r)\, \Big[\mathrm{p}\big(\varepsilon_n + t - r, W_r, z\big) - \mathrm{p}\big(\varepsilon_{n'} + t - r, W_r, z\big)\Big]^2.$$

Then the claim follows from (37). We also got the stated variance formula.

3° (*absolute continuity*) The absolute continuity statement is a consequence of the Basic Lemma 2.7.1 in Dawson and Fleischmann [DF95]. ■

3.7 The occupation measure process Y^K

If the branching rate functional K belongs to $\mathbf{K}^\xi$ for some $\xi > 0$, then the SBM $X = X^K$ has continuous paths by Theorem 17 (b). In this case we can certainly define pathwise the (weighted) *occupation measure process* $Y = Y^K$ related to X by

$$\langle Y_t, \psi\rangle_{[s,t]} := \int_s^t \mathrm{d}r\, \big\langle X_r, \psi(r,\cdot)\big\rangle, \quad 0 \le s \le t, \;\; \psi \in \mathcal{B}^{p,[0,t]}, \;\; \mu \in \mathcal{M}_p\,, \;\; P_{s,\mu}\text{–a.s.}$$

Note that Y_t with respect to $P_{s,\mu}$ is a measure on $[s,t] \times \mathsf{R}^d$. From Proposition 11 (b) we immediately get the *expectation formula*

$$P_{s,\mu} Y_t(\mathrm{d}r\,\mathrm{d}b) \;=\; \mathbf{1}_{[s,t]}(r)\,\mathrm{d}r\, S_{r-s}\,\mu(\mathrm{d}b) \in \mathcal{M}_p^{[s,t]}, \tag{38}$$

$0 \le s \le t,\;\; \mu \in \mathcal{M}_p\,,\;\; b \in \mathsf{R}^d$. It is routine to show that the bivariate process $[X,Y]$ satisfies Hypothesis 12 with

$$f_1(s,a) = \Pi_{s,a}\varphi(W_t), \quad f_2(s,a) = \Pi_{s,a}\int_s^t \mathrm{d}r\; \psi(r,W_r). \tag{39}$$

Moreover, by Lemma 13 and Remark 14, it has centered moments of all orders which satisfy the recursive schema (29). In particular, by setting $\theta_1 = 0$ and $\theta_2 = 1$, the following *variance formula* holds (recall Remark 14):

$$V\mathrm{ar}_{s,\mu}\,\langle Y_t, \psi\rangle_{[s,t]} \;=\; 2\,\Pi_{s,\mu}\int_s^t K(\mathrm{d}r)\Big[\int_r^t \mathrm{d}\sigma\;\Pi_{r,W_r}\,\psi(\sigma,W_\sigma)\Big]^2, \tag{40}$$

$0 \le s \le t,\;\; \mu \in \mathcal{M}_p\,,\;\; \psi \in \mathcal{B}^{p,[0,t]}$.

3.8 Occupation times with absolutely continuous states

Introduce the *occupation time*

$$Y_{[s',t]} = Y^K_{[s',t]} := \int_{s'}^t \mathrm{d}r\; X^K_r\,, \qquad 0 \le s \le s' \le t, \tag{41}$$

of the interval $[s',t]$, related to the process X^K distributed according to $P_{s,\mu}$, $\mu \in \mathcal{M}_p$. Opposed to Y_t from the previous subsection, $Y_{[s',t]}$ is a measures on R^d. We also need the (inhomogeneous) Brownian *potential kernel*

$$\mathrm{q}(s',t,a,b) = \mathrm{q}(s',t,b-a) := \int_{s'}^t \mathrm{d}r\; \mathrm{p}(r,a,b), \qquad 0 \le s' \le t, \quad a,b \in \mathsf{R}^d. \tag{42}$$

Write

$$\mu * \mathrm{q}\,(s',t,b) := \int \mu(\mathrm{d}a)\,\mathrm{q}(s',t,a,b), \qquad \mu \in \mathcal{M}_p\,, \quad 0 \le s' \le t, \quad b \in \mathsf{R}^d.$$

Proposition 23 (absolutely continuous states of $Y^K_{[s',t]}$) *Fix $\mu \in \mathcal{M}_p$ and $0 \le s \le s' \le t$. If $s = s'$, additionally suppose that*

$$\mu * \mathrm{q}\,(0,r,z) \quad \textit{is (finite and) continuous in} \quad r \ge 0, \tag{43}$$

for each $z \in \mathsf{R}^d$. Assume that $K \in \mathbf{K}^*$ satisfies

$$\Pi_{s,\mu}\int_s^t K(\mathrm{d}r)\,\mathrm{q}^2\Big(\varepsilon + r', \varepsilon' + r', W_r\,, z\Big) \;\xrightarrow[0<\varepsilon\le\varepsilon'\downarrow 0]{}\; 0, \qquad z \in \mathsf{R}^d, \tag{44}$$

for both $r' = (s'-r)^+$ *and* $r' = t-r$. *Then the* $L^2(P_{s,\mu})$ *-limit of*

$$y^{\varepsilon}_{[s',t]}(z) := \left\langle Y_{[s',t]}\,,\, \mathrm{p}(\varepsilon,\cdot,z)\right\rangle, \qquad \varepsilon > 0, \tag{45}$$

exists as $\varepsilon \downarrow 0$, *and is denoted by* $y_{[s',t]}(z) = y^K_{[s',t]}(z)$, *for each* $z \in \mathsf{R}^d$. *Hence, with respect to* $P_{s,\mu}$, *the random measure* $Y_{[s',t]}$ *on* R^d *is absolutely continuous with density function* $y_{[s',t]}$:

$$P_{s,\mu}\Big(Y_{[s',t]}(\mathrm{d}z) = y_{[s',t]}(z)\,\mathrm{d}z\Big) = 1. \tag{46}$$

The following moment formulas hold:

$$\begin{aligned} P_{s,\mu}\, y_{[s',t]}(z) &= \mu * \mathrm{q}\,(s'-s, t-s, z), \\ Var_{s,\mu}\, y_{[s',t]}(z) &= \Pi_{s,\mu} \textstyle\int_s^t K(\mathrm{d}r)\ \mathrm{q}^2\big((s'-r)^+, t-r, W_r\,, z\big) < +\infty, \end{aligned} \tag{47}$$

$z \in \mathsf{R}^d$.

$y_{[s',t]}(z) = y^K_{[s',t]}(z)$ is called the (weighted) *occupation density* (*super-Brownian local time*) of X^K (with respect to $P_{s,\mu}$) at z during the time interval $[s',t]$. This proposition can be applied, for instance, in the single point catalyst model to restate partly a result of [DF94].

Proof 1° (*convergence of expectations*) First of all, by (38) we get for the expectation of (45):

$$P_{s,\mu}\, y^{\varepsilon}_{[s',t]}(z) = \mu * \mathrm{q}\,(\varepsilon + s' - s, \varepsilon + t - s, z), \qquad z \in \mathsf{R}^d. \tag{48}$$

By the assumption (43), we conclude that (48) (written as an appropriate difference) converges to $\mu * \mathrm{q}(s'-s, t-s, z)$. This is the desired limiting expectation as it occurs in (47).

2° (*Cauchy sequence*) We want to show that $y^{\varepsilon}_{[s',t]}(z)$ is Cauchy in $L^2(P_{s,\mu})$ as $\varepsilon \downarrow 0$. Because of Step 1°, it suffices to prove that

$$Var_{s,\mu}\left[y^{\varepsilon}_{[s',t]}(z) - y^{\varepsilon'}_{[s',t]}(z)\right] \underset{\varepsilon,\varepsilon'\downarrow 0}{\longrightarrow} 0.$$

Inserting the definitions (45) and (41), by the variance formula (40) with $\psi(r,\cdot) = \mathbf{1}\{s' \le r \le t\}\,\mathrm{p}(\varepsilon,\cdot,z)$, the variance expression at the l.h.s. equals

$$2\,\Pi_{s,\mu} \textstyle\int_s^t K(\mathrm{d}r)\Big[\mathrm{q}\big(\varepsilon+(s'-r)^+, \varepsilon+t-r, W_r, z\big) - \mathrm{q}\big(\varepsilon'+(s'-r)^+, \varepsilon'+t-r, W_r, z\big)\Big]^2.$$

Assuming $\varepsilon \le \varepsilon'$, this can be estimated from above by four times the sum (concerning the two cases of r') of the l.h.s. expressions in (44). (Note that no finiteness problems appear by the assumptions $K \in \mathbf{K}^*$ and $\varepsilon, \varepsilon' > 0$.) Hence, $y^{\varepsilon}_{[s',t]}(z)$ is Cauchy, therefore the L^2-limit $y_{[s',t]}(z)$ exists, and the moment formulas (47) hold.

3° (*absolute continuity*) The remaining statement (46) follows from the Basic Lemma 2.7.1 of [DF95]. ∎

4 The catalyst process ϱ

$K(\mathrm{d}r) \equiv \gamma\,\mathrm{d}r$ is the most studied and very well understood special case. Here each hidden X-particle branches with the *constant* rate $\gamma > 0$. Then, for smooth $\varphi \in \mathcal{C}_+^p$ and t fixed, the solution $v = v(\cdot, t, \cdot)$ of the cumulant equation (23) uniquely solves even the parabolic equation

$$-\frac{\partial v}{\partial s} = \frac{1}{2}\Delta v - \gamma\, v^2, \qquad v\Big|_{s=t} = \varphi,$$

(differentiate (23) formally using the semigroup of W). The corresponding $\mathcal{M}_p$-valued Markov process $X = X^{\gamma\,\mathrm{d}r}$ (first constructed by Iscoe [Isc86]) is *time-homogeneous.* So without loss of generality we may start it at time $s = 0$. Moreover, by Theorem 17 (b), it is *continuous* (Konno and Shiga [KS88]).

As announced, the particular continuous super-Brownian motion $X = X^{\gamma\,\mathrm{d}r}$ will be used to govern the branching in the catalytic SBM we will introduce in § 5.4. For convenience, *from now on* we write ϱ instead of $X^{\gamma\,\mathrm{d}r}$, and $\mathbb{P}_\mu$ instead of $P_{0,\mu}$ in this case $K(\mathrm{d}r) = \gamma\,\mathrm{d}r$, and call ϱ the *catalyst process.*

If the initial state ϱ_0 of ϱ is even *random*, we write $\mathbb{P}$ for the law of ϱ. But for simplicity then we always impose that $\|\varrho_0\|_p = \langle \varrho_0\,, \phi_p \rangle$ has finite moments of *all* orders, and that the law of ϱ_0 is (spatially) *shift invariant.* (Note that this implies that the constant β in the definition of the reference function ϕ_p is positive.) However, in most cases we assume $\varrho_0 = \ell$ where ℓ is a (not necessarily normalized) *Lebesgue measure* on R^d. Then, of course, $\mathbb{P} = \mathbb{P}_\ell$. Note also that $\mathbb{P}$ covers the case if ϱ_0 is distributed according to an ergodic steady state (in dimensions $d \geq 3$).

The main objective of this section is to establish in dimensions $d \leq 3$ the existence of a *jointly Hölder continuous occupation density field* related to the catalyst process ϱ, if the initial state ϱ_0 is not too irregular (Theorems 31 and 33 at pp. 35 and 36).

4.1 Jointly continuous occupation density field of ϱ

For notational reason, we first restate a result of Sugitani [Sug89]. Recall that q denotes the Brownian potential kernel, and $Y^{\gamma\,\mathrm{d}r}_{[\delta,\delta+t]}$ the occupation time of the interval $[\delta, \delta+t]$, related to $X^{\gamma\,\mathrm{d}r} = \varrho$, introduced in (42) and (41), respectively.

Lemma 24 (jointly continuous occupation density field) *Let $d \leq 3$. Fix $\mu \in \mathcal{M}_p$ and $\delta \geq 0$. If $\delta = 0$, assume additionally that*

$$[r, z] \mapsto \mu * \mathrm{q}(0, r, z) \quad \textit{is continuous on} \quad \mathsf{R}_+ \times \mathsf{R}^d. \tag{49}$$

Then, with respect to $\mathbb{P}_\mu$, there is a (jointly) continuous field

$$y_\delta = y_\delta^{\gamma\,\mathrm{d}r} := \left\{ y_{[\delta,\delta+t]}(z);\ t \geq 0,\ z \in \mathsf{R}^d \right\}$$

on $\mathsf{R}_+ \times \mathsf{R}^d$ *such that for the occupation times* $Y_{[\delta,\delta+t]} = Y^{\gamma\,\mathrm{d}r}_{[\delta,\delta+t]}$ *we have*

$$\mathbb{P}_\mu\Big(Y^{\gamma\,\mathrm{d}r}_{[\delta,\delta+t]}(\mathrm{d}z) = y_{[\delta,\delta+t]}(z)\,\mathrm{d}z \quad \textit{for all} \quad t \geq 0\Big) = 1.$$

y_δ is called the (jointly continuous) *occupation density field* (*super-Brownian local time*) of $Y^{\gamma\,\mathrm{d}r}_{[\delta,\delta+(\cdot)]}$.

The next three subsections contain some preliminaries aimed to the proof of the announced Hölder continuities of y_δ. To carry this out, we use ideas from Sugitani [Sug89], but avoid his "formal" exponential moments and power series arguments, as well as his "identity" (4.6). In fact, an $L^2(\mathbb{P}_\mu)$ - functional of $\{\varrho_t\,;\ t \geq \delta\}$ for $\delta > 0$ will in general depend on (the random) ϱ_δ not only via its expectation $S_\delta\,\mu$. Under way we will give a self-contained proof of Sugitani's joint continuity property in § 4.5.

Note that (49) automatically holds in dimension $d = 1$ ([Sug89, Proposition 3.1]). Also, if $\delta > 0$, then for *any* $\mu \in \mathcal{M}_p$ and dimension d, instead of (49) even the following *"delayed" local Lipschitz continuity statement* holds:

Lemma 25 (delayed local Lipschitz) *Let* $d \geq 1,\ \mu \in \mathcal{M}_p$, *and* $\delta > 0$. *Then* $\mu * \mathrm{q}\,(\delta, \delta + r, z)$ *is locally Lipschitz continuous in* $[r, z] \in \mathsf{R}_+ \times \mathsf{R}^d$. *Moreover, the Lipschitz constants are proportional to* $\|\mu\|_p = \langle \mu, \phi_p \rangle$.

To prepare for the proof, it is useful to expose the following elementary heat flow estimate (cf. [Sug89, (3.17)]).

Lemma 26 (a heat flow estimate) *There is a constant* const *depending on* d *and* β *entering in the definition* (4) *of the reference function* ϕ_p, *such that*

$$\phi_p(z)\,\mu * \mathrm{p}(r, z) \;\leq\; \mathrm{const}\ \|\mu\|_p\, r^{-d/2}\,(1 + r)^{p/2}, \qquad \mu \in \mathcal{M}_p\,, \quad r > 0, \quad z \in \mathsf{R}^d.$$

Proof It suffices to show that

$$\int \mu(\mathrm{d}a)\ \exp -\frac{|z - a|^2}{2r} \;\leq\; \mathrm{const}\ \|\mu\|_p\,(1 + r)^{p/2}\,/\phi_p(z)$$

for any μ, r, z. If we restrict the integral additionally to $|a| \leq 4|z|$, then we can replace the exponential expression by $1 \leq \mathrm{const}\ \phi_p(a)/\phi_p(z)$. On the other hand, if $|a| \geq 4|z|$, then $|a - z|^2 \geq |a|^2/2$. But for $|a| \leq 1$ we estimate $\exp -\frac{|a|^2}{4r}$ from above by $1 \leq \mathrm{const}\ \phi_p(a)$, whereas for $|a| \geq 1$ we bound it by $\leq \mathrm{const}\ (|a|^2/r)^{-p/2} \leq \mathrm{const}\ \phi_p(a)\,(1 + r)^{p/2}$. ■

Proof of Lemma 25 By the mean value theorem,

$$\begin{aligned}
&\big|\mu * \mathrm{q}\,(\delta, \delta + r, z + x) - \mu * \mathrm{q}\,(\delta, \delta + r, z)\big| \\
&\leq\ \int_\delta^{\delta + r} \mathrm{d}s \int \mu(\mathrm{d}a)\,|x|\,\mathrm{p}(s, z + \theta x - a)\,|z + \theta x - a|/s
\end{aligned}$$

for some $\theta = \theta(z, x, a, s) \in [0, 1]$. Split a factor $\exp -\frac{|z+\theta x-a|^2}{4s}$ away from the p-expression. By Lemma 26, its integral with respect to $\mu(\mathrm{d}a)$ is bounded by a multiple of $||\mu||_p$, uniformly in bounded z, x, s. But the remaining integrand is bounded by a constant times $|x|$, since s is bounded away from 0. Hence, we got the local Lipschitz continuity in x *uniformly* in a bounded r, with a Lipschitz constants proportional to $||\mu||_p$.

On the other hand, $\mu * \mathrm{q}(\delta, \delta + r, z)$ is locally Lipschitz continuous in r, *uniformly* in a bounded z, with a Lipschitz constants proportional to $||\mu||_p$, which follows again from Lemma 26. ∎

4.2 Another estimate for the recursive scheme

As a preparation for the Proof of Lemma 24 we need a refinement of Lemma 10 concerning an estimate for the solution of the recurrence schema for derivatives of the cumulant equation for a particular f and in the present constant branching rate situation $K(\mathrm{d}r) = \gamma\, \mathrm{d}r$.

Lemma 27 (estimate related to time increments) *Fix $N > 0$, and let $d \leq 3$. There are constants $c_k > 0$, $k \geq 2$, such that $\limsup_{k\to\infty} c_k^{1/k} < +\infty$, and that the following holds. Fix $0 < \varepsilon \leq N$, $z \in \mathsf{R}^d$, and $0 \leq t \leq t+h \leq N$. In the recurrence schema* (18)*, consider the special case $K(\mathrm{d}r) = \gamma\, \mathrm{d}r$ and*

$$f(s,a) = \mathrm{q}\Big((t-s)^+ + \varepsilon, t+h+\varepsilon-s, a, z\Big), \qquad s \in I = [0, t+h], \quad a \in \mathsf{R}^d.$$

Then, for $k \geq 2$,

$$\left|u^{(k)}(s,a)\right| \;\leq\; k!\, c_k \left(\gamma \sqrt{h+\varepsilon}\,\right)^{k-1} \mathrm{q}\Big((t-s)^+, t+2(h+\varepsilon)-s, a, z\Big), \qquad (50)$$

$0 \leq s \leq t+h,\ a \in \mathsf{R}^d$.

Proof *Step* 1° Trivially,

$$f(s,a) \leq \mathrm{q}\Big((t-s)^+, t+2(h+\varepsilon)-s, a, z\Big). \qquad (51)$$

Then from (18) we get

$$u^{(2)}(s,a) \;\leq\; 2\gamma \int_s^{t+h} \mathrm{d}r\ \Pi_{s,a}\, \mathrm{q}^2\Big((t-r)^+, t+2(h+\varepsilon)-r, W_r\,, z\Big). \qquad (52)$$

We consider two cases for the r-variable in this integral: $r < (t \vee s)$, and the opposite.

Step 2° In the first case, we conclude that $t > s$. Then for this part of (52) we obtain the upper bound

$$\leq\; 2\gamma \int_s^t \mathrm{d}r\ \Pi_{s,a}\, \mathrm{q}\Big(t-r, t+2(h+\varepsilon)-r, W_r\,, z\Big) \int_{t-r}^{t+2(h+\varepsilon)-r} \mathrm{d}\sigma\ (2\pi\sigma)^{-d/2}.$$

But the $\Pi_{s,a}$-expectation of the q-expression equals $\mathrm{q}\big(t-s, t+2(h+\varepsilon)-s, a, z\big)$, and, by the substitution $t-r \mapsto r$, the remaining double integral can be written as

$$\int_0^{t-s} \mathrm{d}r \int_r^{r+2(h+\varepsilon)} \mathrm{d}\sigma \; (2\pi\sigma)^{-d/2} \le \text{const}\,\sqrt{h+\varepsilon}\,, \tag{53}$$

where, in this proof, *const* always refers to a constant depending only on N.

Step 3° In the case $r \ge (t \vee s)$ we first rewrite $\mathrm{q}^2\big(0, t+2(h+\varepsilon)-r, W_r\,, z\big)$ as

$$\int_{(0\,,\,t+2(h+\varepsilon)-r]^2} \mathrm{d}[s_1\,,s_2]\,\big[2\pi(s_1+s_2)\big]^{-d/2}\,\mathrm{p}\Big(\tfrac{s_1 s_2}{s_1+s_2}\,, W_r\,, z\Big). \tag{54}$$

Then in the part of (52) under consideration we interchange the order of integration and get the upper bound

$$\gamma \int_{(0\,,\,t+2(h+\varepsilon)-(t\vee s)]^2} \mathrm{d}[s_1\,,s_2]\,(s_1+s_2)^{-3/2} \int_{t\vee s}^{t+h} \mathrm{d}r\;\mathrm{p}\Big(r-s+\tfrac{s_1 s_2}{s_1+s_2}\,, a, z\Big),$$

except a factor *const.* The inner integral is a q-term, which, by the elementary inequalities $t+2(h+\varepsilon)-(t\vee s) \le 2(\varepsilon+h)$ and

$$0 \le \tfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} \le m \qquad \text{if} \qquad 0 < \sigma_1, \sigma_2 \le 2m, \tag{55}$$

can be estimated from above by $\mathrm{q}\big((t-s)^+, t+2(h+\varepsilon)-s, a, z\big)$. For the remaining integral we use

$$\int_{(0\,,\,2(h+\varepsilon)]^2} \mathrm{d}[s_1\,,s_2]\,(s_1+s_2)^{-3/2} \;\le \text{const}\,\sqrt{h+\varepsilon}\,, \tag{56}$$

since $d \le 3$.

Step 4° Altogether we showed

$$\begin{aligned} &\gamma \textstyle\int_s^{t+h} \mathrm{d}r\;\Pi_{s,a}\,\mathrm{q}^2\Big((t-r)^+, t+2(h+\varepsilon)-r, W_r\,, z\Big) \\ &\quad\le C\,\big(\gamma\sqrt{h+\varepsilon}\,\big)\,\mathrm{q}\Big((t-s)^+, t+2(h+\varepsilon)-s, a, z\Big) \end{aligned} \tag{57}$$

with a constant C depending only on N. Hence, by (52), we get (50) in the case $k=2$ with $c_2 := C$.

Step 5° Define $\{c_k\,;\ k \ge 1\}$ by (21). Then, based on (51) and (57), the claim easily follows by induction on k. ∎

4.3 Further estimates for the recursive scheme

We still need a version of Lemma 27 for another function f.

Lemma 28 (estimate related to space increments) *Let $d \le 3$. Fix numbers $0 \le \delta < N$ and a constant α such that*

$$0 < \alpha < 1 \wedge (2 - d/2). \tag{58}$$

There are constants $c_k > 0$, $k \ge 2$, such that $\limsup_{k\to\infty} c_k^{1/k} < +\infty$, and that the following holds. Fix $0 < \varepsilon, \gamma \le N$, $z_1, z_2 \in \mathsf{R}^d$ with $|z_1|, |z_2| \le N$, and $0 \le t \le \delta + t \le N$. In the recurrence schema (18)*, consider the special case $K(\mathrm{d}r) = \gamma\,\mathrm{d}r$ and*

$$f(s,a) = \mathrm{q}\big(\varepsilon+\delta, \varepsilon+\delta+t-s, a, z_1\big) - \mathrm{q}\big(\varepsilon+\delta, \varepsilon+\delta+t-s, a, z_2\big),$$

$s \in I = [0, \delta+t]$, $a \in \mathsf{R}^d$. Then, for $k \ge 2$,

$$\big|u^{(k)}(s,a)\big| \;\le\; k!\, c_k\, \gamma^{k/2}\, |z_1 - z_2|^{k\alpha}\, Q(s,a), \tag{59}$$

$0 \le s \le \delta + t$, $a \in \mathsf{R}^d$, where

$$Q(s,a) := \textstyle\sum_{i=1}^2 \mathrm{q}\Big((\delta - s)^+, 2\big(\varepsilon+\delta+t-s\big), a, z_i\Big). \tag{60}$$

Proof 1° (*two inequalities*) From [Sug89, (3.43), (3.44)], we borrow the following two inequalities concerning the Brownian transition density:

$$\begin{aligned}\big[\mathrm{p}(r,a) + \mathrm{p}(r,b)\big]\big[\mathrm{p}(s,a) + \mathrm{p}(s,b)\big]& \\ \le 3\,(rs)^{-d/4}\Big[\mathrm{p}\Big(\tfrac{rs}{r+s}, a\Big) + \mathrm{p}\Big(\tfrac{rs}{r+s}, b\Big)\Big]&,\end{aligned} \tag{61}$$

$$\big|\mathrm{p}(r,a) - \mathrm{p}(r,b)\big| \le c_\alpha\, r^{-\alpha/2}\, |a-b|^\alpha \big[\mathrm{p}(2r,a) + \mathrm{p}(2r,b)\big], \tag{62}$$

$r, s > 0$, $a, b \in \mathsf{R}^d$, where c_α is a constant only depending on α.

2° (*initial step of induction*) In virtue of the initial formula in the recurrence schema (18),

$$\begin{aligned}u^{(2)}(s,a) \;\le\; 2\gamma \textstyle\int_s^{t+\delta} \mathrm{d}r\; \Pi_{s,a} \int_{[\varepsilon+\delta\,,\,\varepsilon+\delta+t-r]^2} \mathrm{d}[s_1\,, s_2]& \\ \textstyle\prod_{i=1}^2 \big|\mathrm{p}(s_i\,, W_r\,, z_1) - \mathrm{p}(s_i\,, W_r\,, z_2)\big|&.\end{aligned} \tag{63}$$

We distinguish between two cases for the variable r in the outer integral, namely between $s \le r < (\delta \vee s)$ and the opposite. In the first case, we conclude $0 \le r < \delta \le s_1$, hence, since $\delta > 0$ is fixed,

$$\big|\mathrm{p}(s_1\,, W_r\,, z_1) - \mathrm{p}(s_1\,, W_r\,, z_2)\big| \le \mathrm{const}\; |z_1 - z_2|,$$

where, in this proof, *const* is always a constant only depending on α, δ, and N. For the other factor, related to s_2, we apply (62) and calculate the $\Pi_{s,a}$-expectation. Thus, for the first part of the r.h.s. of (63) we get the bound

$$\text{const}\ \gamma\, |z_1 - z_2|^{2\alpha} \int_s^\delta \mathrm{d}r \int_{\varepsilon+\delta}^{\varepsilon+\delta+t-r} \mathrm{d}s_2\ s_2^{-\alpha/2} \sum_{i=1}^2 \mathrm{p}\big(r - s + 2s_2\,, a, z_i\big)$$

(recall that $|z_i| \le N$). Since $s_2^{-\alpha/2}$ is bounded, the inner integral can be estimated from above by *const* $Q(s,a)$, whereas the remaining integral is bounded by *const*.

Turning to the other case $(\delta \vee s) \le r \le t + \delta$, we apply (62) and (61) to get the upper bound

$$2\,\gamma \int_{\delta \vee s}^{t+\delta} \mathrm{d}r\ \Pi_{s,a} \int_{[\varepsilon+\delta\,,\,\varepsilon+\delta+t-r]^2} \mathrm{d}[s_1\,, s_2]\, c_\alpha^2\, (s_1 s_2)^{-\alpha/2-d/4}$$
$$|z_1 - z_2|^{2\alpha}\, 3\Big[\mathrm{p}\Big(2\,\tfrac{s_1 s_2}{s_1+s_2}\,, W_r\,, z_1\Big) + \mathrm{p}\Big(2\,\tfrac{s_1 s_2}{s_1+s_2}\,, W_r\,, z_2\Big)\Big]$$

for this part of the r.h.s. of (63). Evaluating the expectation, interchanging the order of integration, and using (55) leads to the upper bound

$$\text{const}\ \gamma\, |z_1 - z_2|^{2\alpha} \int_{(0\,,\,2(\varepsilon+\delta+t-s)]^2} \mathrm{d}[s_1\,, s_2]\, (s_1 s_2)^{-\alpha/2-d/4}$$
$$\int_{(\delta-s)^+}^{2(\varepsilon+\delta+t-s)} \mathrm{d}r\ \big[\mathrm{p}(r,a,z_1) + \mathrm{p}(r,a,z_2)\big].$$

(The additional factor 2 in the upper integral bound of the outer integral will be useful later in related calculations of the induction step.) By our assumption (58) on α, the first of these integrals can be absorbed into *const.*

Altogether, we showed

$$u^{(2)}(s,a) \ \le\ 2\, C\, \gamma\, |z_1 - z_2|^{2\alpha}\, Q(s,a)$$

where C is a constant only depending on α, δ, and N. This finishes the initial step of induction.

3° (*induction step*) Define the c_k by (21). Consider $k \ge 3$, and assume that the inequality (59) holds for $2, \ldots, k-1$. Then the two terms of $u^{(k)}(s,a)$ in (18) can be estimated as in the initial step of induction with the obvious modifications (in particular, enlarging C where needed). Here in the case of the first term, under $(\delta \vee s) \le r \le t + \delta$, we use (55) with $[\sigma_1\,, \sigma_2] = [2s_1\,, s_2]$. On the other hand, for the second term, under $s \le r < (\delta \vee s)$, apply

$$\mathrm{q}\Big(\delta - r, 2(\varepsilon+\delta+t-r), W_r\,, z_i\Big) \le \text{const} \int_{\delta-r}^{2(\varepsilon+\delta+t-r)} \mathrm{d}s_1\ s_1^{-3/2} \le \text{const}\ (\delta - r)^{-1/2}$$

which is integrable with respect to $\mathrm{d}r$ on the interval $(0, \delta)$.

This then finishes the proof altogether. ■

4.4 Moment estimates related to the occupation density

With respect to $\mathbb{P}_\mu$, $\mu \in \mathcal{M}_p$, as in (45), set

$$y^{\varepsilon}_{[\delta,\delta+t]}(z) = \big\langle Y_{[\delta,\delta+t]}\,,\, \mathrm{p}(\varepsilon,\cdot,z)\big\rangle, \qquad 0 \le \delta \le \delta+t,\ \varepsilon > 0,\ z \in \mathsf{R}^d, \tag{64}$$

for the approximate (if $\varepsilon > 0$ is small) occupation density at z during the time interval $[\delta, \delta+t]$. By the expectation formula (48),

$$Z^{\delta,\varepsilon}_t(z) \;:=\; \mu * \mathrm{q}\big(\varepsilon+\delta, \varepsilon+\delta+t, z\big) - y^{\varepsilon}_{[\delta,\delta+t]}(z), \qquad t \ge 0,\ z \in \mathsf{R}^d, \tag{65}$$

is the related centered field. Recall that γ is the constant branching rate.

Lemma 29 (moment estimates for time increments of $Z^{\delta,\varepsilon}$) *Fix $N > 0$, and let $d \le 3$. Then to each $k \ge 2$ there exists a constant c_k such that*

$$\Big|\mathbb{P}_\mu\big[Z^{\delta,\varepsilon}_{t+h}(z) - Z^{\delta,\varepsilon}_t(z)\big]^k\Big| \le c_k\, \big(\gamma\sqrt{h+\varepsilon}\big)^{k/2} \sum_{i=1}^{k-1} \Big[\mu * \mathrm{q}\Big(\delta+t, \delta+t+2(h+\varepsilon), z\Big)\Big]^i,$$

$\mu \in \mathcal{M}_p$, $0 < \varepsilon, \gamma \le N$, $z \in \mathsf{R}^d$, *and* $0 \le \delta \le \delta+t \le \delta+t+h \le N$.

Proof Setting

$$\theta_1 = 0, \quad \theta_2 = 1, \quad \psi(r,a) \equiv \mathbf{1}_{[\delta+t,\delta+t+h]}(r)\, \mathrm{p}(\varepsilon, a, z),$$

and recalling f_2 from (39) (with t replaced by $\delta+t+h$), we may identify Z from Lemma 13 (with $s = 0$ and $f = f_2$) with the present $Z^{\delta,\varepsilon}_{t+h}(z) - Z^{\delta,\varepsilon}_t(z)$ of the lemma. In particular, the corresponding moments satisfy the recursive schema (29). Then, based on Lemma 27 (with t replaced by $\delta+t$), we can proceed by induction using (32). ∎

Lemma 30 (moment estimates for space increments of $Z^{\delta,\varepsilon}$) *Let $d \le 3$, and fix $0 \le \delta < N$ as well as a constant α satisfying* (58). *Then to each $k \ge 2$ there exists a constant c_k such that*

$$\Big|\mathbb{P}_\mu\big[Z^{\delta,\varepsilon}_t(z_1) - Z^{\delta,\varepsilon}_t(z_2)\big]^k\Big| \le c_k\, \gamma^{k/2}\, |z_1 - z_2|^{k\alpha} \sum_{i=1}^{k-1} \Big[\mu * \sum_{j=1}^{2} \mathrm{q}\big(\delta, 2(\varepsilon+\delta+t), z_j\big)\Big]^i,$$

$\mu \in \mathcal{M}_p$, $0 < \varepsilon, \gamma \le N$, $z_1, z_2 \in \mathsf{R}^d$, $|z_1|, |z_2| \le N$, *and* $0 \le t \le \delta+t \le N$.

Proof Setting

$$\theta_1 = 0, \quad \theta_2 = 1, \quad \psi(r,a) \equiv \mathbf{1}_{[\delta,\delta+t]}(r)\, \big[\mathrm{p}(\varepsilon, a, z_1) - \mathrm{p}(\varepsilon, a, z_2)\big],$$

and recalling f_2 from (39) (with t replaced by $\delta+t$), we may identify Z from Lemma 13 (with $s = 0$ and $f = f_2$ and recalling Remark 14) with the present $Z^{\delta,\varepsilon}_t(z_1) - Z^{\delta,\varepsilon}_t(z_2)$ of the lemma. In particular, the corresponding moments satisfy the recursive schema (29). Then, based on Lemma 28, we can again proceed by induction. ∎

4.5 Proof of Sugitani's joint continuity result

The purpose of this subsection is to *prove Lemma* 24. Let $d \leq 3$. Fix $\delta \geq 0$ and $\mu \in \mathcal{M}_p$ satisfying (49) if $\delta = 0$.

1° (*existence of* $y_{[\delta,\delta+t]}(z)$) According to Proposition 23 at p. 24, for the existence of the $L^2(\mathbb{P}_\mu)$ - limit $y_{[\delta,\delta+t]}(z)$ of $y^\varepsilon_{[\delta,\delta+t]}(z)$ (defined in (64)) as $\varepsilon \downarrow 0$, it suffices to show that for $t \geq 0$ and $z \in \mathsf{R}^d$ fixed,

$$\Pi_{0,\mu} \int_0^{\delta+t} \mathrm{d}r\ \mathrm{q}^2\Big(\varepsilon + r', \varepsilon' + r', W_r\,, z\Big) \xrightarrow[0<\varepsilon\leq\varepsilon'\downarrow 0]{} 0, \tag{66}$$

for both $r' = (\delta - r)^+$ and $r' = \delta + t - r$.

In the case $r' = (\delta - r)^+$, rewrite q^2 as in (54), and calculate the expectation over the new p-term to get

$$\int_0^{\delta+t} \mathrm{d}r \int_{(\varepsilon+(\delta-r)^+,\varepsilon'+(\delta-r)^+]^2} \mathrm{d}[s_1, s_2]\, \tfrac{1}{(s_1+s_2)^{d/2}}\, \mu * \mathrm{p}\Big(r + \tfrac{s_1 s_2}{s_1+s_2}\,, z\Big), \tag{67}$$

except a constant factor.

Consider first the $r \geq \delta$ part of this integral. Interchanging the order of integration, and using the elementary inequality (55) to get rid of $\frac{s_1 s_2}{s_1+s_2}$ in the new $\mu * \mathrm{q}$-expression, results in the bound

$$\int_{(\varepsilon,\varepsilon']^2} \mathrm{d}[s_1, s_2]\, (s_1 + s_2)^{-d/2}\ \mu * \mathrm{q}\,(\delta, \delta + t + 1, z)$$

(assuming $\varepsilon' \leq 1$). By the joint continuities (49) and Lemma 25 (i), this $\mu * \mathrm{q}$-expression is finite, and an estimate as in (56) shows that we are dealing with a negligible term as $\varepsilon' \downarrow 0$.

If $\delta > 0$, we still have to handle the $r \leq \delta$ part of (67), and estimate this to

$$\int_0^{\delta} \mathrm{d}r \int_{(\varepsilon+\delta-r,\varepsilon'+\delta-r]^2} \mathrm{d}[s_1, s_2]\, \big[(s_1 + s_2)r + s_1 s_2\big]^{-3/2} \int \mu(\mathrm{d}a) \exp -\frac{\frac{1}{2}\,|z-a|^2}{r + \frac{s_1 s_2}{s_1+s_2}}\,,$$

except a constant factor. Since $r + \frac{s_1 s_2}{s_1+s_2}$ is bounded (recall (55)), the inner integral can be bounded by a constant. If we additionally assume that $r \geq \frac{\delta}{2}$, then we may estimate the integrand to $(s_1 + s_2)^{-3/2}$ (except a constant). By monotonicity in s_1 and s_2 of this new integrand, the remaining inner integral can be estimated from above by integrating over $(\varepsilon, \varepsilon']^2$. By (56), this results also in a negligible term. If on the other hand we restrict to $r \leq \frac{\delta}{2}$, then the integrand can be replaced by $(s_1 s_2)^{-3/2}$. Again by monotonicity, we now may integrate over $(\varepsilon + \delta/2, \varepsilon' + \delta/2]^2$, ending up again in an error term.

Altogether, (67) tends to 0 as $\varepsilon' \downarrow 0$.

Concerning the other case $r' = \delta + t - r$, in (66) substitute $r \mapsto \delta + t - r$, estimate one q-factor by *const* $\int_\varepsilon^{\varepsilon'} \mathrm{d}s_1\, (r + s_1)^{-d/2}$, calculate the expectation of the other q-factor to get $\mu * \mathrm{q}(\varepsilon+\delta+t, \varepsilon'+\delta+t, z)$ converging to 0 as $\varepsilon' \downarrow 0$ by applying twice the delayed local Lipschitz Lemma 25 (i). The remaining double integral also tends to zero.

Summarizing, (66) holds, establishing the existence of $y_{[\delta,\delta+t]}(z)$.

2° (*centering*) Since $\mathbb{P}_\mu\, y_{[\delta,\delta+t]}(z) = \mu * \mathrm{q}(\delta, \delta+t, z)$ by the expectation formula in (47), which is jointly continuous by (49) and Lemma 25 (i), it suffices to show that the *centered* field

$$Z_t^\delta(z) := \mu * \mathrm{q}(\delta, \delta+t, z) - y_{[\delta,\delta+t]}(z) \tag{68}$$

has a jointly continuous modification.

3° (*moment estimates*) Fix $N > 0$ and $k \geq 1$. Then, based on Lemma 29, we have the following *moment estimates of time increments:*

$$\mathbb{P}_\mu \big| Z_{t+h}^\delta(z) - Z_t^\delta(z) \big|^{2k} \;\leq\; c_k \left(\gamma \sqrt{h}\right)^k \sum_{i=1}^{2k-1} \Big[\mu * \mathrm{q}\big(\delta+t, \delta+t+2h, z\big) \Big]^i, \tag{69}$$

$0 \leq t \leq \delta+t \leq \delta+t+h \leq N,\;\; z \in \mathsf{R}^d$. (Recall that γ is the branching rate in the model we are just discussing.) In fact, by our joint continuity assumption (49), by Lemma 25 (i), and by the L^2-convergence of Step 1°, the l.h.s. can be estimated from above by

$$\liminf_{\varepsilon \downarrow 0} \mathbb{P}_\mu \big| Z_{t+h}^{\delta,\varepsilon}(z) - Z_t^{\delta,\varepsilon}(z) \big|^{2k}$$

with $Z_t^{\delta,\varepsilon}(z)$ from (65). Similarly, based on Lemma 30, for a fixed α satisfying (58), the following *moment estimates of space increments* hold:

$$\begin{aligned} &\mathbb{P}_\mu \big| Z_t^\delta(z_1) - Z_t^\delta(z_2) \big|^{2k} \\ &\leq\; c_k\, \gamma^k \left| z_1 - z_2 \right|^{2k\alpha} \textstyle\sum_{i=1}^{2k-1} \Big[\mu * \sum_{j=1}^2 \mathrm{q}\big(\delta, 2(\delta+t), z_j\big) \Big]^i, \end{aligned} \tag{70}$$

$|z_1|, |z_2|, \gamma \leq N,\;\; 0 \leq t \leq \delta+t \leq N.$

4° (*conclusion*) Choosing a number k sufficiently large, the existence of a jointly continuous version of Z^δ follows from (69) and (70) by Kolmogorov's moment criterion, since the sums at the r.h.s. are finite by the continuity assumption (49) and the delayed Lipschitz Lemma 25 (i).

This finishes the proof. ■

4.6 Hölder continuous occupation densities

In the previous subsection we used the moment estimates (69) and (70) for the construction of a jointly continuous modification of the occupation density field y_δ. But actually they even imply the existence of a jointly *Hölder* continuous modification. (Recall that $\|\mu\|_p = \langle \mu, \phi_p \rangle$.)

Theorem 31 (jointly Hölder continuous occupation density field) *Fix $d \leq 3,\ \xi \in \big(0, \frac{1}{4}\big)$, and let k denote the smallest natural number satisfying*

$$k > \tfrac{2(d+1)}{1-4\xi}\,. \tag{71}$$

Fix $\mu \in \mathcal{M}_p$ and $\delta \geq 0$. If $\delta = 0$, assume additionally that

$$\begin{aligned}[r,z] \mapsto \mu * \mathrm{q}(0,r,z) \quad &\textit{is locally } \xi\textit{−Hölder continuous on } \mathsf{R}_+ \times \mathsf{R}^d \\ &\textit{with Hölder constants proportional to } \|\mu\|_p = \langle \mu, \phi_p \rangle\,.\end{aligned} \tag{72}$$

Then, with respect to $\mathbb{P}_\mu$, there exists a modification of the occupation density field $y_\delta = y_\delta^{\gamma\,\mathrm{d}r} = \big\{y_{[\delta,\delta+t]}(z);\ t \geq 0,\ z \in \mathsf{R}^d\big\}$ related to $Y^{\gamma\,\mathrm{d}r}_{[\delta,\delta+(\cdot)]}$, such that, with $\mathbb{P}_\mu$-probability one, for each $N \geq 1$,

$$\big|y_{[\delta,\delta+t_1]}(z_1) - y_{[\delta,\delta+t_2]}(z_2)\big| \;\leq\; C_{\xi,N,k}\big|[t_1,z_1] - [t_2,z_2]\big|^{\xi}, \tag{73}$$

$[t_i,z_i] \in E_N := [0,N] \times [-N,+N]^d,\ i = 1,2$. Here $C_{\xi,N,k}$ is a random constant with a finite moment of order $2k$ (with respect to $\mathbb{P}_\mu$) satisfying

$$\mathbb{P}_\mu C^{2k}_{\xi,N,k} \leq \mathrm{const}\,\big(1 \vee \|\mu\|_p^{2k}\big) \tag{74}$$

with the constant const *independent of μ.*

In particular, the occupation density field y_δ is locally (jointly) ξ-Hölder continuous.

Proof Fix d,μ,δ,ξ,N as in the theorem, $0 < \gamma \leq N$, and $\alpha > \frac{1}{4}$ satisfying (58). By this choice of α, using the triangular inequality we can combine the moment estimates (69) and (70) for the centered field Z^δ (defined in (68)) related to y as follows:

$$\mathbb{P}_\mu\Big(c_k^{-\frac{1}{2k}}\Big[Z^\delta_{t_1}(z_1) - Z^\delta_{t_2}(z_2)\Big]\Big)^{2k} \;\leq\; \Big|[t_1,z_1] - [t_2,z_2]\Big|^{k/2}, \tag{75}$$

$[t_i,z_i] \in E_N,\ i = 1,2$. Here, by the delayed local Lipschitz Lemma 25, c_k is a non-random polynomial in $\|\mu\|_p$ of at most order $2k$. By the definition of k we have

$$\xi < \tfrac{k/2\,-\,(1+d)}{2k}\,. \tag{76}$$

Therefore, by Theorem 1.2.1 of [RY91] there is a modification of Z^δ, for simplicity again denoted by Z^δ, such that

$$c_k^{-\frac{1}{2k}}\big|Z^\delta_{t_1}(z_1) - Z^\delta_{t_2}(z_2)\big| \;\leq\; C\big|[t_1,z_1] - [t_2,z_2]\big|^{\xi}, \qquad [t_i,z_i] \in E_N,\ \ i = 1,2.$$

Here C is a random constant with finite $\mathbb{P}_\mu$-moment of order $2k$, and from the proof there we conclude that this moment does not depend on the concrete distribution entering into the moment estimate (75), hence on μ. Thus,

$$\big|Z^\delta_{t_1}(z_1) - Z^\delta_{t_2}(z_2)\big| \;\leq\; C_k\big|[t_1,z_1] - [t_2,z_2]\big|^{\xi}, \qquad [t_i,z_i] \in E_N,\ \ i = 1,2.$$

with $\mathbb{P}_\mu C_k^{2k} \le$ const $c_k \le$ const $\left(1 \vee \|\mu\|_p^{2k}\right)$. From the triangular inequality, combined with assumption (72) and the delayed local Lipschitz Lemma 25 (i), the statements (73) and (74) immediately follow. ■

Remark 32 (Hölder in the space variable) If we fix a time point $t \ge 0$ and ask for regularity only in the space variable, then the analogous statements of the theorem hold even for $\xi \in \left(0, \frac{1}{2}\right)$, provided that (72) holds for such a ξ. In particular, then $y_{[\delta,\delta+t]}(z)$ is locally ξ-Hölder continuous in z. (In fact, for the choice of α, impose additionally $\alpha > \xi$, in order to guarantee the existence of a k with the required property (76). ◇

For our purpose, we later actually need a different version of the Hölder property of y_δ and in fact under $\mathbb{P}$. Recall that we agreed that $\mathbb{P}$ means that ϱ starts with a random ϱ_0 having a shift-invariant law and finite moments of all orders of $\|\varrho_0\|_p$. In the following theorem, $\mathbb{P} = \mathbb{P}_\ell$ (with ℓ a Lebesgue measure) satisfies the assumption in the case $\delta = 0$, whereas $\delta > 0$ is aimed to the steady state case in dimension $d = 3$.

Theorem 33 (Hölder in time with some space uniformity) *Fix $d \le 3$, $\xi \in \left(0, \frac{1}{4}\right)$, $\delta \ge 0$, and $\mathbb{P}$. If $\delta = 0$, assume additionally that (72) holds with μ replaced by ϱ_0, with $\mathbb{P}$-probability one. Then, with respect to $\mathbb{P}$, there is a modification y_δ of the occupation density field such that for each $N \ge 1$*

$$\sup_{\substack{0 \le t_1, t_2 \le N \\ z_1, z_2 \in \mathsf{R}^d \\ [t_1, z_1] \ne [t_2, z_2]}} \frac{\left|y_{[\delta,\delta+t_1]}(z_1)\,\phi_p(z_1) \;-\; y_{[\delta,\delta+t_2]}(z_2)\,\phi_p(z_2)\right|}{\left|[t_1\,, z_1] - [t_2\,, z_2]\right|^{\xi}} \tag{77}$$

is finite $\mathbb{P}$-a.s.

Proof Fix $d, \xi, \delta, \mathbb{P}$ as in the theorem, and let k denote the smallest natural number satisfying (71). Write $H(B)$ for the supremum expression in (77) if there R^d is replaced by a Borel set $B \subset \mathsf{R}^d$. It suffices to show that

$$\mathbb{P}\big(H(\mathsf{R}^d) > T\big) \underset{T\to\infty}{\longrightarrow} 0. \tag{78}$$

Now

$$\mathbb{P}\big(H(\mathsf{R}^d) > T\big) \;\le\; \textstyle\sum_m \mathbb{P}\big(H(C_m) > T\big)$$

where $C_m := [0,1)^d + m,\ m \in \mathsf{Z}^d$. This inequality can be continued with

$$\le \sum_m \mathbb{P}\left(\sup_{\substack{0 \le t_1, t_2 \le N \\ z_1, z_2 \in C_m \\ [t_1, z_1] \ne [t_2, z_2]}} \frac{\left|y_{[\delta,\delta+t_1]}(z_1) \;-\; y_{[\delta,\delta+t_2]}(z_2)\right|}{\left|[t_1\,, z_1] - [t_2\,, z_2]\right|^{\xi}} \;>\; \frac{\text{const}\ T}{\phi_p(m)} \right). \tag{79}$$

By Theorem 31, the supremum expression can be bounded from above by a random constant $C_{\xi,N,m.k}$. Hence, using Tchebychev's inequality, for (79) we get the upper bound

$$\text{const}\ T^{-2k}\sum_m \left(\phi_p(m)\right)^{2k} \mathbb{P}C^{2k}_{\xi,N,m,k}\,. \tag{80}$$

By the shift-invariance of $\mathbb{P}$, the latter $2k$-th moment expression is independent of m, and by (74) it can be bounded from above by *const* $\mathbb{P}\|\varrho_0\|_p^{2k}$, which is finite by assumption. Hence, (78) follows, finishing the proof. ∎

Remark 34 (choice of the Hölder index) The choices of the Hölder indexes ξ in the Theorems 31,33 and Remark 32 are adapted to the dimension $d = 3$. For $d < 3$, these choices are *not* optimal. In fact, here ξ can grow by the factor 2. Moreover, in dimension one, $y_{[\delta,\delta+t]}(z)$ is even differentiable in t since ϱ has a jointly continuous density (see Proposition 44 below). ◇

5 Brownian collision local times

The main purpose of the section is to construct the Brownian collision local time $L = L[W, \varrho]$ for $\mathbb{P}_\ell$-almost all catalyst process paths ϱ, see Theorem 40 at p. 40. (Recall ℓ denotes a Lebesgue measure.)

As in the finite measure case of Evans and Perkins [EP94], we proceed in two steps. First we construct $L = L[W, \eta]$ for an appropriate (deterministic) $\mathcal{M}_p$-valued path η, we call regular: Proposition 37 (a) below is a version of Evans and Perkins [EP94, Theorem 4.1] adapted for our needs. In the second step (§ 5.3), we then verify that in dimensions $d \le 3$ the catalyst process ϱ has regular $\mathcal{M}_p$-valued paths, a.s. with respect to $\mathbb{P}_\ell$. Here our methods differ from those of [EP94] in that we replace estimates of the uniform modulus of continuity bound by some Hölder continuity properties of the occupation density field $y := y_{[0,\,\cdot\,]}$ of ϱ, established in Theorem 33, which for us seems to be a more natural approach.

5.1 Preparation: Regular $\mathcal{M}_p$-valued paths η

For the moment, fix $N > 0$, $\varepsilon \in (0, 1]$, and η in the set $\mathcal{C}[\mathsf{R}_+, \mathcal{M}_p]$ of all *continuous* $\mathcal{M}_p$-valued paths. Set

$$h(\eta, \varepsilon, N) := \sup_{0\le s\le N,\ a\in\mathsf{R}^d} \int_s^{s+\varepsilon} \mathrm{d}r \left\langle \eta_r\ ,\ \phi_p\, \mathrm{p}(r - s, a, \cdot)\right\rangle. \tag{81}$$

(Recall that p denotes the Brownian transition density.) Heuristically,

$$\left\langle \eta_r\ ,\ \phi_p \mathrm{p}(r - s, a, \cdot)\right\rangle = \left\langle \phi_p\eta_r\ ,\ \mathrm{p}(r - s, a, \cdot)\right\rangle$$

approaches the "density" $\langle \phi_p \eta_s , \delta_a \rangle$ at a of the finite measure $\phi_p(b)\eta_s(\mathrm{d}b)$ on R^d as $r \downarrow s$. Note however, that in the cases we are mostly interested in, these "densities" degenerate. But in (81) we have an additional integration with respect to $\mathrm{d}r$. Hence, intuitively, the integral in (81) measures the *"ε-accumulated density"* of $\phi_p\eta$ at $[s,a]$.

Definition 35 (regular $\mathcal{M}_p$-valued paths) For our purpose, a path η in $\mathcal{C}[\mathsf{R}_+ , \mathcal{M}_p]$ is called *regular* if $h(\eta,\varepsilon,N) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0$ for all $N > 0$ (and h defined in (81)). $\diamond$

Roughly speaking, η is regular, if the ε-accumulated densities of the finite measure-valued path $\phi_p\eta$ disappear as $\varepsilon \downarrow 0$, uniformly on $[0,N] \times \mathsf{R}^d$, for each $N > 0$.

Example 36 (one-dimensional regular paths) For $d = 1$, *all* continuous $\mathcal{M}_p$-valued paths η are regular. In fact,

$$\sup_{s\le N,\ a\in\mathsf{R}} \int_s^{s+\varepsilon} \mathrm{d}r \left\langle \eta_r \ , \ \phi_p \, \mathrm{p}(r-s,a,\cdot) \right\rangle \le \text{const}\,\sqrt{\varepsilon} \sup_{t\le N+1} \left\langle \eta_t , \phi_p \right\rangle \underset{\varepsilon\downarrow 0}{\longrightarrow} 0,$$

for each $N > 0$. $\diamond$

5.2 Brownian collision local time $L[W,\eta]$ of a regular η

Recalling Definition 35, fix a regular path η, and $\varepsilon \in (0,1]$. Define a continuous additive functional $L^\varepsilon = L^\varepsilon[W,\eta]$ of the Brownian motion W by

$$L^\varepsilon(\mathrm{d}r) := \left\langle \eta_r , \mathrm{p}(\varepsilon, W_r, \cdot) \right\rangle \mathrm{d}r. \tag{82}$$

We interpret L^ε as the collision local time of η with the *ε-vicinity* of the Brownian path W. Now we are prepared to introduce the Brownian collision local time $L = L[W,\eta]$ of a regular η, which at the same time satisfies all requirements of a branching rate functional (recall Definition 1).

Proposition 37 (Brownian collision local time $L[W,\eta]$ of η) *Let η be a regular ($\mathcal{M}_p$-valued) path. Then there exists an additive functional $L = L[W,\eta]$ of the Brownian motion W with the following properties.*

(a) (existence of L) *Let ψ be a (strictly) positive function in $\mathcal{C}^{\,p,[0,N]}$ (defined in § 2.1), for $N > 0$. Then*

$$\sup_{0\le s\le N,\ a\in\mathsf{R}^d} \Pi_{s,a} \sup_{s\le t\le N} \left| \int_s^t L^\varepsilon(\mathrm{d}r)\,\psi(r,W_r) - \int_s^t L(\mathrm{d}r)\,\psi(r,W_r) \right|^2 \underset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

(b) (expectation of L) *For measurable $\psi : \mathsf{R}_+ \times \mathsf{R}^d \mapsto \mathsf{R}_+$, and $0 \le s \le t$, as well as $a \in \mathsf{R}^d$,*

$$\Pi_{s,a} \int_s^t L(\mathrm{d}r)\,\psi(r,W_r) \;=\; \int_s^t \mathrm{d}r \int \eta_r(\mathrm{d}b)\,\mathrm{p}(r-s,a,b)\,\psi(r,b). \tag{83}$$

(c) (branching rate functional) *This functional L belongs to* $\mathbf{K}$.

Definition 38 (Brownian collision local time $L[W,\eta]$ of η) If η is a regular ($\mathcal{M}_p$-valued) path, then the additive functional $L[W,\eta]$ according to Proposition 37 is called the *Brownian collision local time (BCLT) of η.* $\diamond$

Example 39 (η with jointly continuous density) If the regular path η has a *jointly continuous density field* on $(0,+\infty)\times\mathsf{R}^d$, that is $\eta_r(\mathrm{d}b)=\eta_r(b)\mathrm{d}b$ with $[r,b]\mapsto\eta_r(b)$ continuous (as the catalyst process ϱ has in dimension $d=1$, see Proposition 44 below), then

$$L(\mathrm{d}r)=\eta_r(W_r)\mathrm{d}r \quad \Pi_{s,a}\text{−a.s.},\quad a\in\mathsf{R}^d. \tag{84}$$

$\diamond$

Proof of Proposition 37 **(a)** First of all note that $t\mapsto\psi(t,a)\eta_t(\mathrm{d}a)$ is a continuous $\mathcal{M}_f$-valued path on $[0,N]$, for simplicity denoted by $\psi\eta$. For $\varepsilon\in(0,1]$, define a related continuous additive functional $A^\varepsilon=A^\varepsilon[W,\psi\eta]$ as in (82), with η replaced by $\psi\eta$. Recall that by the assumed regularity of η,

$$\sup_{s\le N,\ a\in\mathsf{R}^d}\int_s^{s+\varepsilon}\mathrm{d}r\int\eta_r(\mathrm{d}b)\,\psi(r,b)\,\mathrm{p}(r-s,a,b)\underset{\varepsilon\downarrow 0}{\longrightarrow}0,\qquad N>0.$$

Then by a simple modification of Evans and Perkins [EP94, Theorem 4.1], there is a continuous additive functional $A=A[W,\psi\eta]$ of W such that

$$\sup_{s\le N,\ a\in\mathsf{R}^d}\Pi_{s,a}\sup_{s\le t\le N}\left|A^\varepsilon(s,t)-A(s,t)\right|^2\underset{\varepsilon\downarrow 0}{\longrightarrow}0,\qquad N>0. \tag{85}$$

(The modification, we apply without further notice, consists in replacing their assumption that $\int\eta_t(\mathrm{d}b)\,\psi(t,b)$ vanishes for t sufficiently large by restricting the consideration to t running in finite intervals $[0,N]$.) Then we set $L(\mathrm{d}r):=A(\mathrm{d}r)/\psi(r,W_r)$, and the existence statement (a) is obviously fulfilled.

(b) For each fixed $N>0$, by monotone convergencies, we may restrict our considerations to ψ satisfying additionally the assumptions in (a). Then from (a) we get the pointwise convergence

$$\Pi_{s,a}\int_s^t L^\varepsilon(\mathrm{d}r)\,\psi(r,W_r)\underset{\varepsilon\downarrow 0}{\longrightarrow}\Pi_{s,a}\int_s^t L(\mathrm{d}r)\,\psi(r,W_r).$$

By definition, the expectation at the l.h.s. equals

$$\int_s^t\mathrm{d}r\int\eta_r(\mathrm{d}b)\,\mathrm{p}(\varepsilon+r-s,a,b)\,\psi(r,b)\,.$$

But it is easy to show that this double integral converges to the r.h.s. of (83) as $\varepsilon\downarrow 0$, which is finite by our assumptions that ψ is dominated and η is regular. (For instance, take two-sided estimates of $\mathrm{p}(\varepsilon+r-s,a,b)$, and monotone limits.)

(c) Claim (c) is an immediate consequence of the expectation formula (83), applied to $\psi=\psi_p$, and the assumed regularity of η. This finishes the proof. $\blacksquare$

5.3 BCLT of the catalyst process ϱ in dimensions $d \leq 3$

Now we are prepared to state the *main result* of this section. Recall that $\mathbb{P}$ refers to the catalyst process starting with a spatial homogeneous initial state ϱ_0 such that $\|\varrho_0\|_p$ has finite moments of all orders. Note that $\mathbb{P} = \mathbb{P}_\ell$ satisfies the assumption in the following theorem with $\delta = 0$, whereas the case $\delta > 0$ covers the ergodic time-stationary $\mathbb{P}$ in dimension 3.

Theorem 40 (Brownian collision local time of the catalyst process) *Fix* $d \leq 3,\ \xi \in \left(0, \frac{1}{4}\right),\ \delta \geq 0,$ *and* $\mathbb{P}$. *If* $\delta = 0$, *assume additionally that* (72) *holds with* μ *replaced by* ϱ_0, *with* $\mathbb{P}$*-probability one. Then* $\mathbb{P}$*-almost surely, the Brownian collision local time* $L = L\big[W, \varrho_{\delta+(\cdot)}\big]$ *exists and is a branching rate functional in* $\mathbf{K}^\xi$.

Proof 1° (*existence*) By Proposition 37 (a), for the existence of the BCLT $L = L[W, \varrho]$ it suffices to show that the paths $\varrho_{\delta+(\cdot)}$ are regular with $\mathbb{P}$-probability one. For this purpose, fix $N > 0,\ \varphi \in \mathcal{B}_+^p$, and, as in Definition 35, look at

$$\int_s^{s+\varepsilon} \mathrm{d}r \int \varrho_{\delta+r}(\mathrm{d}b)\, \varphi(b)\, \mathrm{p}(r-s, a, b), \qquad 0 \leq s \leq s+\varepsilon \leq N,\ \ a \in \mathsf{R}^d. \tag{86}$$

Consider Sugitani's occupation density field $y_\delta = \big\{y_{[\delta,\delta+t]}(z);\ t \geq 0,\ z \in \mathsf{R}^d\big\}$ related to the catalyst process ϱ (Lemma 24). Since $y_{[\delta,\delta+t]}(z)$ is non-decreasing in t, for each fixed $z \in \mathsf{R}^d$, it determines a locally finite (random) measure $\lambda_\delta^z(\mathrm{d}t)$ on R_+. Then (86) can be rewritten as

$$\int \mathrm{d}b\ \varphi(b) \int_s^{s+\varepsilon} \lambda_\delta^b(\mathrm{d}r)\, \mathrm{p}(r-s, a, b). \tag{87}$$

Denoting by $\dot{\mathrm{p}}$ the time derivative of p, we can use the elementary inequalities

$$\mathrm{p}(r-s, a, b)\ \leq\ \mathrm{p}(\varepsilon, a, b) + \int_r^{s+\varepsilon} \mathrm{d}\sigma\ \big|\dot{\mathrm{p}}(\sigma - s, a, b)\big|$$

and

$$\big|\dot{\mathrm{p}}(t, a, b)\big|\ \leq\ \text{const}\ t^{-1}\, \mathrm{p}\big(t, \tfrac{b-a}{2}\big), \qquad t > 0,\ \ a, b \in \mathsf{R}^d,$$

to estimate the interior integral in (87) from above by

$$\mathrm{p}(\varepsilon, a, b)\, \lambda_\delta^b\big([s, s+\varepsilon]\big) + \text{const} \int_s^{s+\varepsilon} \lambda_\delta^b(\mathrm{d}r) \int_r^{s+\varepsilon} \mathrm{d}\sigma\ (\sigma - s)^{-1}\, \mathrm{p}\big(\sigma - s, \tfrac{b-a}{2}\big).$$

Interchanging the order of integration, we can continue with

$$\leq\ \mathrm{p}(\varepsilon, a, b)\, y_{[\delta+s,\, \delta+s+\varepsilon]}(b) + \text{const} \int_s^{s+\varepsilon} \mathrm{d}\sigma\ (\sigma - s)^{-1}\, \mathrm{p}\big(\sigma - s, \tfrac{b-a}{2}\big)\, y_{[\delta+s,\, \delta+\sigma]}(b).$$

But by Theorem 33, $\sup_b y_{[\delta+s,\,\delta+s+\varepsilon]}(b)\phi_p(b) \leq \text{const}\ \varepsilon^{\xi}$, almost surely with respect to P (with a random constant *const*). Using the fact that the db-integration of the Brownian transition density functions equals one, altogether, for the expression in (87) we get the estimate

$$\leq\ \text{const}\Big[\varepsilon^{\xi} + \int_0^{\varepsilon} \mathrm{d}\sigma\ \sigma^{\xi-1}\Big].$$

We therefore obtain

$$\int_s^{s+\varepsilon} \mathrm{d}r \int \varrho_{\delta+r}(\mathrm{d}b)\,\varphi(b)\,\mathrm{p}(r-s,a,b)\ \leq\ \text{const}\ \varepsilon^{\xi}, \tag{88}$$

$0 \leq s \leq s+\varepsilon \leq N,\ \ a \in \mathsf{R}^d$, that is uniformly in the considered s,a. Thus we get the desired regularity of $\varrho_{\delta+(\cdot)}$ according to Definition 35, hence P-almost surely the existence of the BCLT $L = L\big[W, \varrho_{\delta+(\cdot)}\big]$.

2° (*branching rate functional in* $\mathbf{K}^{\xi}$) To complete the proof, according to Definition 3 (b), it suffices to show that

$$\Pi_{s,a} \int_s^{s+\varepsilon} L(\mathrm{d}r)\,\phi_p^2(W_r)\ \leq\ \text{const}\ \varepsilon^{\xi}\,\phi_p(a), \tag{89}$$

$0 \leq s \leq s+\varepsilon \leq N,\ a \in \mathsf{R}^d$. By the expectation formula in Proposition 37 (b), the expectation at the l.h.s. can be written as in (86) with $\varphi = \phi_p^2$. Now proceed as in Step 1° with the only difference to replace the db-integration over the Brownian transition density functions by the domination property (11) of the heat flow to get out the $\phi_p(a)$ needed for (89). This finishes the proof. ■

In dimension $d = 1$ the statement of Theorem 40 can be sharpened. In fact, as already noticed in Example 36, here *all* $\mathcal{M}_p$-valued paths are regular. Hence, by Proposition 37 (a), in dimension one the BCLT $L[W,\varrho]$ exists P_{μ}-a.s., for *all* $\mu \in \mathcal{M}_p$. Moreover (recalling Definition 3 (b)):

Lemma 41 (one-dimensional BCLT $L[W,\varrho]$) *In dimension $d = 1$, for all μ in $\mathcal{M}_p$ and P_{μ}-almost all paths ϱ, the Brownian collision local time $L = L[W,\varrho]$ belongs to $\mathbf{K}^{\xi}$ with $\xi = \frac{1}{2}$.*

Proof We have to show (89). For this aim, consider (86) with $\varphi = \phi_p^2$ and $\delta = 0$. If we restrict additionally to $|b-a| \leq |a|/2$, then $|b| \geq |a|/2$, hence $\phi_p(b) \leq \text{const}\ \phi_p(a)$, and we can use the inequality in Example 36 to arrive at the r.h.s. of (89). In the opposite case $|b-a| > |a|/2$, apply

$$\mathrm{p}(r-s,a,b)\,\phi_p(b)\ \leq\ \mathrm{p}(r-s,a/2)\ \leq\ \text{const}\ (r-s)^{-1/2}\,\phi_p(a)$$

instead, to finish the proof. ■

5.4 Existence: Catalytic SBM X^ϱ for $d \leq 3$

From now on we always *assume* that $d \leq 3$ and consider the catalyst process ϱ distributed by $\mathbb{P}$ which is *assumed* to be either $\mathbb{P}_\ell$ (with ℓ an Lebesgue measure) or an ergodic time-stationary law in dimension $d = 3$. Note that in the latter case $\varrho_{\delta+(\cdot)}$ is again distributed by $\mathbb{P}$, for each $\delta > 0$. Hence, by Theorem 40, in both cases, the BCLT $L = L[W, \varrho]$ exists $\mathbb{P}$-a.s. and is a branching rate functional K in $\mathbf{K}^\xi$, for all $\xi < \frac{1}{4}$. Now we have together all ingredients to define the catalytic SBM rigorously:

Definition 42 (catalytic SBM) If the branching rate functional K is $\mathbb{P}$-a.s. given by the BCLT $L = L[W, \varrho]$ of ϱ, then the we write X^ϱ for the continuous SBM X^K according to Theorem 17 (b), and $P^\varrho_{s,\mu}$, $s \geq 0$, $\mu \in \mathcal{M}_p$, for the *quenched* distributions of X^ϱ given ϱ. We call X^ϱ the *catalytic SBM in the catalytic medium* ϱ distributed by $\mathbb{P}$. $\diamond$

Remark 43 (arbitrary ϱ_0 in $d = 1$) Based on Lemma 41, the catalytic SBM X^ϱ is also well-defined if in Definition 42 we replace $\mathbb{P}$ by $\mathbb{P}_\mu$, $\mu \in \mathcal{M}_p$, provided that $d = 1$. $\diamond$

Recall that by Lemma 15, $X = X^\varrho$ has finite moments of all orders (given ϱ). As a preparation for later usage, we want to expose here only the *covariance formula*. Indeed, by the expectation formula (83) for the BCLT of ϱ, from Proposition 11 (b) we get:

$$\begin{aligned} &\mathcal{C}\mathrm{ov}^\varrho_{s,\mu}\left[\, \langle X_{t_1}, \varphi\rangle, \langle X_{t_2}, \psi\rangle \,\right] \\ &= 2 \int \mu(\mathrm{d}a) \int_s^{t_1 \vee t_2} \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \mathrm{p}(r-s, a, b)\; S_{t_1 - r}\varphi\,(b)\; S_{t_2 - r}\psi\,(b), \end{aligned} \tag{90}$$

$0 \leq s \leq t_1, t_2$, $\mu \in \mathcal{M}_p$ and $\varphi, \psi \in \mathcal{B}^p_+$.

6 Persistence of X^ϱ in dimension one

The main purpose of this section is to study the long-term behavior of the catalytic SBM X^ϱ in the *one-dimensional* case. Recall that under $d = 1$ one can start the catalyst process ϱ with any measure in $\mathcal{M}_p$ (Lemma 41). The main results will be the persistence Theorems 47 and 48 at pp. 46 and 49. For convenience, we first restate the existence of a jointly continuous density field of ϱ.

6.1 Jointly continuous density of the catalyst process ϱ

The following result goes back to Konno and Shiga [KS88] and Reimers [Rei89]:

Lemma 44 (jointly continuous density in one dimensions) *In dimension $d=1$ and for all initial measures $\mu \in \mathcal{M}_p$, the catalyst process has a jointly continuous density field, for simplicity also denoted by ϱ :*

$$\mathsf{P}_\mu\Big(\varrho_t(\mathrm{d}z) = \varrho_t(z)\,\mathrm{d}z \quad \text{for all} \quad t > s\Big) = 1, \qquad s \geq 0. \tag{91}$$

Proof $1°$ (*finite initial measure*) If μ is a finite measure, for a proof we refer to Konno and Shiga [KS88].

$2°$ (*decomposition of ϱ*) If B is a Borel subset of R, write ϱ^B for the catalyst process starting with the *restricted* measure $\varrho_0(\,\cdot\,\cap B)$. Decompose R with the help of the intervals $C_m = [m, m+1)$, $m \in \mathsf{Z}$. By the branching property we get the representation $\varrho = \sum_m \varrho^m$ with conditionally independent $\varrho^m := \varrho^{C_m}$, $m \in \mathsf{Z}$, given ϱ_0; see [DP91, Chapter 6].

$3°$ (*Borel-Cantelli*) Let $\mu \in \mathcal{M}_p$. Decompose ϱ with this initial measure μ as in the previous step. Then, for each bounded Borel set $B \subset \mathsf{R}$ there is a constant *const* such that

$$\mathsf{P}_\mu\Big(\varrho_t^m(B) > 0 \quad \text{for some } t > 0\Big) \;\leq\; \text{const}\ |m|^{-2}, \qquad m \geq 2,$$

see Iscoe [Isc88, Theorem 1]. Since this is summable in m, with P_μ-probability one only finitely many ϱ^m will ever have mass in B, and by the Steps $1°$ and $2°$ the jointly continuous density field exists also in this infinite measure case. ■

Consider now the (one-dimensional) catalytic SBM X^ϱ with ϱ distributed by P_μ, $\mu \in \mathcal{M}_p$. Note that in this case, for smooth $\varphi \in \mathcal{C}_+^p$ and t fixed, the solution $v = v(\cdot, t, \cdot)$ of the cumulant equation (23) uniquely solves the (one-dimensional) parabolic equation

$$-\frac{\partial v}{\partial s} \;=\; \frac{1}{2}\,\Delta v - \varrho\, v^2, \qquad v\Big|_{s=t} = \varphi, \tag{92}$$

with ϱ the jointly continuous density function distributed according to P_μ (recall (91)). Notice also that v then satisfies the following *"Feynman-Kac equation"* (that is, Feynman-Kac version of the cumulant equation (92))

$$v(s,t,a) \;=\; \Pi_{s,a}\, \varphi(W_t)\, \exp - \int_s^t \mathrm{d}r\; \varrho_r(W_r)\, v(r,t,W_r), \tag{93}$$

$0 \leq s \leq t$, $a \in \mathsf{R}$, we will later use. (In fact, start for instance with Dynkin [Dyn94b, Theorem 4.2.1], and use the fact that each non-negative continuous function on $[0,t] \times \mathsf{R}$ can pointwise be approximated from below by bounded smooth functions, and proceed by two-sided estimates and monotone limits as in the proof of (13).)

6.2 Finite time of interference in dimension one

If the catalyst process ϱ starts with a finite initial mass, $\|\mu\| < \infty$, then by the extinction property of Feller's critical branching diffusion, ϱ has only a finite life time: $\varrho_t = 0$ for all t sufficiently large, with P_μ-probability one. On the other hand, if $\|\mu\| = \infty$, then (in the present one-dimensional situation), one still has a *local extinction:* For each bounded Borel set $B \subset \mathsf{R}$,

$$\varrho_t(B) = 0 \quad \text{for all } t \text{ sufficiently large, } \mathsf{P}_\mu\text{–a.s., } \mu \in \mathcal{M}_p\,.$$

Intuitively, in dimension $d = 1$, at a late time t, the catalyst process forms clumps of huge mass but each has (spatially) bounded support which will not intersect a given region. At time t, neighboring clumps have a "distance" of order t ([DF88]), whereas a tagged Brownian path W has a "range" of order $\sqrt{t}$. Thus, both ϱ and W will *"interference"* only in a finite time. In other words, the intersection of the graphs of the density field ϱ (recall (91)) and W is bounded:

Proposition 45 (finite time of interference) *Let $d = 1$, $\mu \in \mathcal{M}_p$ with $1 < p < 2$ (for instance $\mu = \ell$), and $s \geq 0$, $b \in \mathsf{R}$. Then there exists a (non-Markovian) random time $\tau \geq s$ such that for the jointly continuous density field ϱ of the catalyst process,*

$$\varrho_t(W_t) = 0 \quad \textit{for all} \quad t \geq \tau, \quad \Pi_{s,b} \times \mathsf{P}_\mu\text{–}a.s.$$

Proof 1° (*decomposition and Borel-Cantelli*) Fix μ, s, b as in the proposition. Decompose $\varrho = \sum_m \varrho^m$ as in Step 2° of the proof of Lemma 44. By the extinction property of Feller's critical branching diffusion, there are stopping times τ^m (life time of ϱ^m) such that

$$\tau^m < \infty, \quad \varrho_t^m = 0 \text{ for } t \geq \tau^m, \quad \mathsf{P}_\mu\text{–a.s.}$$

We may set $\tau := s \vee \sup_m \tau^m$ ones we know that the number of those $m \neq 0$ such that the event

$$E^m := \Big\{ \varrho_t^m(W_t) > 0 \quad \text{for some} \quad t \geq 1 \vee s \Big\}$$

occurs, is finite with $\Pi_{s,b} \times \mathsf{P}_\mu$-probability one. For this, by Borel-Cantelli it suffices to show that

$$\textstyle\sum_{m \neq 0} \Pi_{s,b} \times \mathsf{P}_\mu(E^m) < \infty.$$

2° (*law of the iterated logarithm*) Fix $\varepsilon \in \big(0, (2-p)/p\big)$. From the law of iterated logarithm for Brownian motion we know that we can find a (random) constant $c = c(W) > 0$ such that

$$|W_r| \leq c\, r^{(1+\varepsilon)/2}, \qquad r \geq 1 \vee s, \quad \Pi_{s,b}\text{–a.s.}$$

Now E^m implies one of the following events:

$$E_1^m := \Big\{ \varrho_t^m \big(B(0,|m|/2\big) > 0 \quad \text{for some} \quad t \geq 1 \Big\}$$
$$E_2^m := \Big\{ \varrho_t^m(\mathsf{R}) > 0 \quad \text{for some} \quad t \geq \big(\tfrac{|m|}{2c}\big)^{2/(1+\varepsilon)} \Big\}.$$

That is, ϱ^m has to charge the centered ball of radius $|m|/2$ after time 1, or it has to survive by time $\big(\frac{|m|}{2c}\big)^{2/(1+\varepsilon)}$. Again by Iscoe [Isc88, Theorem 1],

$$\mathbb{P}_\mu(E_1^m) \leq \text{const } |m|^{-2}, \qquad m \neq 0,$$

which is summable in $m \neq 0$. On the other hand, from the well-known survival probability estimate of Feller's critical branching diffusion,

$$\mathbb{P}_\mu(E_2^m) \;\leq\; \mu(C_m)\,\big(\tfrac{|m|}{2c}\big)^{-2/(1+\varepsilon)}, \qquad m \neq 0,$$

(with $C_m = [m, m+1)$), which is also summable in $m \neq 0$, since $2/(1+\varepsilon) \geq p$ and $\mu \in \mathcal{M}_p$ by assumption. This completes the proof. ∎

6.3 Variance of the total BCLT in one dimension

By Proposition 45, in the one-dimensional case the time of interference of W and ϱ is finite $\mathbb{P}_\ell$-a.s. Hence, also the *total* BCLT $L(0,+\infty)$ of ϱ is finite. Actually, even its second moment with respect to $\Pi_{0,b}$ is finite:

Proposition 46 (finite 2nd moment of the total BCLT) *Let $d = 1$. Fix $s \geq 0$ and $b \in \mathsf{R}$. Then the Brownian collision local time $L = L[W,\varrho]$ of ϱ satisfies*

$$\Pi_{s,b} L^2(s,+\infty) = \Pi_{s,b} \Big(\int_s^\infty \mathrm{d}r \; \varrho_r(W_r) \Big)^2 < +\infty \quad \mathbb{P}_\ell\text{--}a.s. \tag{94}$$

Proof By homogeneity properties of W and ϱ, and since Lebesgue measure ℓ is shift-invariant, without loss of generality we may restrict to the case $b = 0$.

1° (*finite time interval*) First of all,

$$\Pi_{s,0} L^2(s,K) < +\infty \quad \mathbb{P}_\ell\text{ –a.s.}, \quad K > s \vee 1.$$

In fact, calculate the $\mathbb{P}_\ell$-expectation:

$$\mathbb{P}_\ell \Pi_{s,0} L^2(s,K) = 2 \int_s^K \mathrm{d}r \int_r^K \mathrm{d}t \, \Pi_{s,0} \, \mathbb{P}_\ell \, \varrho_r(W_r) \varrho_t(W_t). \tag{95}$$

From the covariance formula (90), for $\mu \in \mathcal{M}_p$ we get

$$\begin{aligned} &\mathcal{C}\mathrm{ov}_\mu\big[\varrho_r(W_r), \varrho_t(W_t)\big] \\ &= 2\gamma \textstyle\int \mu(\mathrm{d}a) \int_0^t \mathrm{d}u \int \mathrm{d}b \; \mathrm{p}(u, b-a)\mathrm{p}(r-u, W_r - b)\mathrm{p}(t-u, W_t - b). \qquad (96) \\ &\leq \text{const } \sqrt{t} \; \textstyle\int \mu(\mathrm{d}a) \, \mathrm{p}(r, W_r - a). \end{aligned}$$

Hence, (95) can be estimated from above by const $K^{5/2} < \infty$.

2° (*time interval* $[s+2^n,\ s+2^{n+1})$) For a fixed $n \geq 0$, we decompose ϱ as in Step 2° of proof of Lemma 44, but only in two terms: $\varrho = \varrho^1 + \varrho^2$ where ϱ^1 starts with μ^1 which by definition is ℓ restricted to $I_n^1 := \{a \in \mathsf{R};\ |a| \geq n2^{n/2}\}$, and ϱ^2 starts with the "complementary" measure, namely ℓ restricted to $I_n^2 := \mathsf{R}\backslash I_n^1$. In the first case, as in (95) we pass to the $\mathbb{P}_\ell$-expectation and use the estimate (96) to get

$$\leq \text{const} \int_{s+2^n}^{s+2^{n+1}} \mathrm{d}r \int_r^{s+2^{n+1}} \mathrm{d}t\, \sqrt{t} \int_{|a| \,\geq\, n2^{n/2}} \ell(\mathrm{d}a)\, \mathrm{p}(2r-s,a). \qquad (97)$$

Since s is fixed, using the elementary inequality

$$\int_T^\infty \mathrm{d}a\ \mathrm{p}(t,a) \leq \frac{t}{T}\, \mathrm{p}(t,T), \qquad t,T>0,$$

we can estimate (97) from above by $\leq$ const $2^{5n/2} \exp\left[-\text{ const } n^2\right]$ with two positive constants *const*. But the latter expression is summable in n. Hence the process starting with μ^1 gives a finite contribution to the second moment of the total BCLT.

Now we turn to the second case. By the cluster representation (see [Daw93, Corollary 11.5.3]), $\{\varrho_r^2;\ s+2^n \leq r < s+2^{n+1}\}$ may be generated by a Poissonian number of clusters at time $r = s+2^n$, namely with expectation $\ell(I_n^2)(s+2^n)^{-1} \leq$ const $n2^{-n/2}$. Hence, this Poissonian number is different from 0 with probability bounded from above by const $n2^{-n/2}$. Since this is summable in n, by Borel-Cantelli only finitely many n produce a contribution to $L^2(s+1,+\infty)$. Exploiting Step 1° repeatedly, the proof is finished. ■

6.4 Persistence of the total mass process ($d = 1$)

In the finite measure-valued SBM with constant branching rate, the total mass process (which is Feller's critical branching diffusion) dies a.s. in a finite time. In the single point catalytic SBM (meaningful only in dimension $d = 1$), the total mass process converges to 0 a.s. as time tends to infinity (see Fleischmann and Le Gall [FL95, Corollary 5]). In contrast to both cases, for the one-dimensional X^ϱ we prove a.s. convergence of the total mass process with preservation of the mean (*persistence*) and with a *non-degenerate* limit (non-zero finite variance).

Theorem 47 (total mass persistence) *Let $d = 1$. For $\mathbb{P}_\ell$-almost all ϱ the following holds. Fix $\mu \in \mathcal{M}_f$ and $s \geq 0$. Then*

$$m^\varrho := \lim_{t\to\infty} \|X_t^\varrho\| \quad \textit{exists } P_{s,\mu}^\varrho\text{–}a.s.$$

where the limiting (total) mass m^ϱ has the Laplace function

$$P_{s,\mu}^\varrho \exp -\theta m^\varrho \;=\; \exp -\langle \mu, u_\theta(s)\rangle, \qquad \theta \geq 0, \qquad (98)$$

with $u_\theta \geq 0$ satisfying the Feynman-Kac identity

$$u_\theta(s,a) = \theta\, \Pi_{s,a} \exp - \int_s^\infty \mathrm{d}r\ \varrho_r(W_r)\, u_\theta(r, W_r), \qquad s \geq 0, \quad a \in \mathsf{R}. \tag{99}$$

The (conditional) law of m^ϱ is infinitely divisible and has the following expectation and variance

$$P^\varrho_{s,\mu}\, m^\varrho = \|\mu\|, \quad \mathrm{Var}^\varrho_{s,\mu}\, m^\varrho = 2\, \Pi_{s,\mu} \int_s^\infty \mathrm{d}r\ \varrho_r(W_r) < +\infty \tag{100}$$

(which are non-zero if $\mu \neq 0$).

Proof Recall that by Lemma 41 for $\mathbb{P}_\ell$-a.a. ϱ the BCLT $L = L[W, \varrho]$ is a branching rate functional in $\mathbf{K}^\xi$, for $\xi = \frac{1}{2}$. For the following proof we fix such a ϱ.

1° (*a.s. convergence*) The $P^\varrho_{s,\mu}$-a.s. convergence of $\|X^\varrho_t\|$ to a limit m^ϱ with expectation bounded by $\|\mu\|$ follows from Proposition 20.

2° (*definition of u_θ*) For the given ϱ, the Laplace transform of $\|X^\varrho_t\|$ satisfies

$$P^\varrho_{s,\mu} \exp -\theta\, \|X^\varrho_t\| \;=\; \exp - \int \mu(\mathrm{d}a)\, v(s,t,a), \qquad \theta \geq 0, \tag{101}$$

with v solving the Feynman-Kac equation

$$v(s,t,a) \;=\; \theta\, \Pi_{s,a} \exp - \int_s^t \mathrm{d}r\ \varrho_r(W_r)\, v(r,t,W_r) \tag{102}$$

(recall (93)). In the particular case $\mu = \delta_a$, from the almost sure convergence explained in Step 1°, we conclude for the existence of the finite limit $\lim_{t\to\infty} v(s,t,a) =: u_\theta(s,a)$.

3° (*limiting cumulant equation*) We want to establish that u_θ satisfies the identity (99). First note that the integral at the r.h.s. of (102) can be written as

$$\int_s^t \mathrm{d}r\ \mathbf{1}\{r \leq \tau\}\, \varrho_r(W_r)\, v(r,t,W_r) \tag{103}$$

with τ the finite time of interference from Proposition 45. By bounded convergence (applied to the finite measure $\mathbf{1}\{r \leq \tau\}\, \varrho_r(W_r)\, \mathrm{d}r$ and to the integrands bounded by θ), the integral (103) converges to

$$\int_s^\tau \mathrm{d}r\ \varrho_r(W_r)\, u_\theta(r, W_r)$$

as $t \to \infty$. Again by bounded convergence, the r.h.s. of (102) tends to the one of (99). Consequently, u satisfies the Feynman-Kac equation (99). Again by bounded convergence, the Laplace function formula (98) follows.

$4°$ (*reduction to* $\mu = \delta_a$) Since $P^{\varrho}_{s,\mu}\|X^{\varrho}_t\| = \int\mu(\mathrm{d}a)\,P^{\varrho}_{s,a}\|X^{\varrho}_t\|$, where we abbreviated $P^{\varrho}_{s,a} := P^{\varrho}_{s,\delta_a}$, and because the latter integrand equals 1, by bounded convergence it suffices to show the expectation formula in (100) for $\mu = \delta_a$, $a \in \mathsf{R}$. Similarly, by (90),

$$Var^{\varrho}_{s,\mu}\|X^{\varrho}_t\| = \int\mu(\mathrm{d}a)\,Var^{\varrho}_{s,a}\|X^{\varrho}_t\|,$$

and

$$Var^{\varrho}_{s,a}\|X^{\varrho}_t\| = 2\Pi_{s,a}L(s,t)\ \uparrow\ 2\Pi_{s,a}\int_s^{\infty}\mathrm{d}r\ \varrho_r(W_r) \quad \text{as } t\uparrow+\infty$$

where the latter expectation is finite by Proposition 46. Hence, by monotone convergence it suffices to show the variance formula in (100) for $\mu = \delta_a$.

$5°$ (*expectation*) First of all, from the Feynman-Kac identity (99) we conclude $u_\theta(r,a) \le \theta$, and (98) implies $P^{\varrho}_{s,a}\exp -\theta m^{\varrho} \ge \mathrm{e}^{-\theta}$. Hence, taking the logarithm of (98) and differentiating with respect to θ gives

$$0 \le u'_\theta(r,a) := \frac{\partial}{\partial\theta}u_\theta(r,a) = \mathrm{e}^{u_\theta(r,a)}P^{\varrho}_{r,a}m^{\varrho}\mathrm{e}^{-\theta m^{\varrho}} \le \mathrm{e}^{\theta}P^{\varrho}_{r,a}m^{\varrho} \le \mathrm{e}^{\theta}, \qquad (104)$$

$\theta > 0$. On the other hand,

$$u_\theta(r,a)\big|_{\theta=0+} = 0, \quad u'_\theta(r,a)\big|_{\theta=0+} = P^{\varrho}_{r,a}m^{\varrho} \le 1. \qquad (105)$$

Next consider

$$w_\theta(s,a) := \Pi_{s,a}\mathrm{e}^{-\int_s^{\infty}\mathrm{d}r\ \varrho_r(W_r)u_\theta(r,W_r)}\int_s^{\infty}\mathrm{d}r\ \varrho_r(W_r)\,u'_\theta(r,W_r). \qquad (106)$$

Of course, the exponential expression is bounded by 1, whereas for the derivative term we use the bound (104). Therefore

$$\limsup_{\theta\downarrow0} w_\theta(s,a) \le \Pi_{s,a}\int_s^{\infty}\mathrm{d}r\ \varrho_r(W_r) = \Pi_{s,a}L(s,+\infty) < +\infty \qquad (107)$$

by Proposition 46.

From (105) and differentiating (99) with respect to θ at $\theta = 0+$, we obtain the following expectation formula:

$$P^{\varrho}_{s,a}m^{\varrho} = 1 - \lim_{\theta\downarrow0}\theta\,w_\theta(s,a)$$

with w_θ from (106). But the latter limit expression disappears by (107), and we arrive at the claimed expectation formula $P^{\varrho}_{s,a}m^{\varrho} \equiv 1$. Moreover,

$$\lim_{\theta\downarrow0}u'_\theta(r,a) = 1 \quad \text{uniformly in } r,a, \qquad (108)$$

and by dominated convergence we get the following sharpening of (107):

$$\lim_{\theta\downarrow 0} w_\theta(s,a) = \Pi_{s,a} L(s,+\infty) < +\infty. \tag{109}$$

6° (*variance*) From the Laplace function (98) we have

$$Var^{\varrho}_{s,a}\, m^{\varrho} = -u''_\theta(s,a)\big|_{\theta=0+}.$$

Differentiating twice (99) we get three terms. The first one is $-2w(s,a)$ with w from (106) and results by (109) into the desired variance expression in (100). The second one is θ times a term as w but with the square of the latest integral in (106). By (104) and Proposition 46, this is a negligible term. Finally, the third term differs from (106) by having $u''_\theta(r,W_r)$ instead of $u'_\theta(r,W_r)$ (except the factor θ). Differentiating (104), we obtain the estimate

$$\left|u''_\theta(r,W_r)\right| \le \mathrm{e}^{2\theta} + \mathrm{e}^{\theta} P^{\varrho}_{r,W_r}(m^{\varrho})^2 \le \mathrm{e}^{2\theta} + \mathrm{e}^{\theta}\left[1 + 2\Pi_{r,W_r} L(r,+\infty)\right].$$

Inserting into the third term and using the Markov property of W, again a second moment expression occurs which is finite by Proposition 46. Altogether we showed the variance formula as desired, finishing the proof. ■

6.5 Persistence in the infinite measure case ($d = 1$)

Starting X^{ϱ} with a Lebesgue measure, opposed to other one-dimensional spatial branching processes, here the catalytic SBM X^{ϱ} does *not* become locally extinct and is even *persistent:*

Theorem 48 (one-dimensional persistence) *In dimension $d = 1$, for $\mathbb{P}_\ell$-almost all realizations ϱ of the catalyst process, the catalytic SBM X^{ϱ} satisfies:*

$$X^{\varrho}_t \underset{t\to\infty}{\longrightarrow} \ell \quad P^{\varrho}_{s,\ell} - \textit{stochastically}, \qquad s \ge 0,$$

(in the p-vague topology).

Proof Fix a catalyst process realization ϱ such that $L = L[W,\varrho]$ is a branching rate functional in $\mathbf{K}^{\xi}$ for $\xi = \frac{1}{2}$ (which by Lemma 41 is possible with $\mathbb{P}_\ell$-probability one). Fix also $s \ge 0$ and a smooth $\varphi \in \mathcal{C}_+$ with compact support. It suffices to show that

$$-\log P^{\varrho}_{s,\ell} \exp -\left\langle X^{\varrho}_t, \varphi\right\rangle \underset{t\to\infty}{\longrightarrow} \langle \ell, \varphi\rangle =: \|\varphi\|_1 .$$

By Proposition 11 (a), the l.h.s. is given by $\|v(s,t)\|_1$, where v solves (23), that is (92). By the *Feynman-Kac* identity (93) and Proposition 45,

$$v(s,t,a) = \Pi_{s,a}\, \varphi(W_t) \exp\Big[-\int_s^{t\wedge\tau} \mathrm{d}r\, \varrho_r(W_r)\, v(r,t,W_r)\Big].$$

Hence,

$$v(r,t,b) \leq \Pi_{r,b}\,\varphi(W_t) \leq (t-r)^{-1/2}\,\|\varphi\|_1 \leq \big((t-\tau)^+\big)^{-1/2}\,\|\varphi\|_1$$

for $r \leq t \wedge \tau$, and we conclude

$$\|v(s,t)\|_1 \geq \Pi_{s,\ell}\,\varphi(W_t)\exp\Big[-\big((t-\tau)^+\big)^{-1/2}\,\|\varphi\|_1\int_s^\tau \mathrm{d}r\,\varrho_r(W_r)\Big].$$

Since the exponential expression converges monotonically and $\Pi_{s,\ell}$-a.s. to one as $t\to\infty$, by dominated convergence the r.h.s. tends to $\Pi_{s,\ell}\,\varphi(W_t) = \|\varphi\|_1$, which also dominates the l.h.s. Hence, $\|v(s,t)\|_1 \underset{t\to\infty}{\longrightarrow} \|\varphi\|_1$, and the proof is finished. ■

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